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Toroidal compactification without vector structure

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Abstract: Many important ideas about string duality that appear in conventional $\mathbf{T}^2$ compactification have analogs for $\mathbf{T}^2$ compactification without vector structure. We analyze some of these issues and show, in particular, how orientifold planes associated with $Sp(n)$ gauge groups can arise from T-duality and how they can be interpreted in F-theory. We also, in an appendix, resolve a longstanding puzzle concerning the computation of $\mathrm{Tr}\,(-1)^F$ in four-dimensional supersymmetric Yang-Mills theory with gauge group $SO(n)$.

Keywords: String and Superstring Scattering, String Duality, M- and F-theories and Other Generalizations.

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Contents

1 Introduction

The gauge group of what is often informally called the $SO(32)$ heterotic string is actually Spin(32)$/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ is generated by an element x of Spin(32) which if projected to $SO(32)$ becomes the generator -1 of the center of $SO(32)$ [1]. In particular, x acts as -1 in the 32 dimensional representation of $SO(32)$, so this representation is not present for the heterotic string. Likewise, x acts as ± 1 for positive and negative chirality spinors of Spin(32), so only the positive chirality spinors are present.

Because particles in the vector (or negative chirality spinor) representation of Spin(32) are absent in the theory, it is possible to consider compactifications of the heterotic string in which the topology of the gauge bundle is such that these representations would actually be impossible. This happens if Dirac quantization is obeyed for $\mathrm{Spin}(32)/\mathbb{Z}_2$ representations, but not for the vector representation of $SO(32)$.

Given a $\mathrm{Spin}(32)/\mathbb{Z}_2$ bundle V over a spacetime manifold M, the obstruction to "vector structure" is measured by a mod two cohomology class $\widetilde{w}_2(M)$, which assigns the value $+1$ to any two-cycle in M on which there is vector structure, and -1 for those for which there is not. The name $\widetilde{w}_2$ is intended to reflect the analogy with the second Stieffel-Whitney class w_2, which is the obstruction to spin structure.[1]

One reason for study of compactification without vector structure is that many of the simplest orientifold constructions, such as K3 models constructed in [2, 3] can be interpreted as compactifications without vector structure. Also, dualities can relate more familiar string compactifications to compactifications without vector structure. This has been seen [4, 5] in studies of compactification on K3 surfaces. The intent of the present paper is primarily to study compactification without vector structure in the more elementary case of compactification on $\mathbf{T}^2$. We will also, in less detail, study certain related and analogous models in other dimensions.

In fact, the model we will focus on has been studied, from a rather different vantage point (not stressing the topology) in [6]. A number of important features were pointed out in that work, including the continuous interpolation from $Sp(8)$ to $SO(16)$ and the role of a discrete theta angle. The model has also been investigated recently in [7] and in [8], which appeared while the present paper was being written and have some overlap with it.

Since the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string and the Type I superstring are equivalent already in ten dimensions, any of its compactifications can be studied in either formalism. This is so whether there is vector structure or not. In addition, for the conventional case with vector structure, there are the following important dualities that arise upon compactification on $\mathbf{T}^2$:

(1) Starting with the Type I superstring on $\mathbf{T}^2$, one can make a T-duality transformation to a Type IIB orientifold on $\mathbf{T}^2/\mathbb{Z}_2$ with 16 sevenbrane pairs and 4 orientifold planes [9, 10].

(2) In the strong coupling limit of the heterotic string, one gets a description via F-theory on K3 [11], which can also be obtained as a strong coupling limit [12] of the orientifold as seen by a threebrane probe [13].

(3) Finally, via a heterotic string T-duality transformation, one can map to a standard $\mathbf{T}^2$ compactification of the $E_8 \times E_8$ heterotic string [14, 15].

[1]A formal definition of $\widetilde{w}_2(V)$ is as follows. Cover M with small open sets U_i on which V is trivial. Let V_{ij} be $\mathrm{Spin}(32)/\mathbb{Z}_2$-valued transition functions on $U_i \cap U_j$, defining V. So in particular $V_{ij}V_{jl}V_{li} = 1$ in $U_i \cap U_j \cap U_l$ for all i, j, and l. Pick a set of liftings $\widetilde{V}_{ij}$ of V_{ij} to $\mathrm{Spin}(32)$. Then $W_{ijk} = \widetilde{V}_{ij}\widetilde{V}_{jk}\widetilde{V}_{ki}$ is equal to ± 1 for all i, j, k (since it equals $+1$ if projected to $\mathrm{Spin}(32)/\mathbb{Z}_2$), obeys $W_{ijk}W_{jkl}W_{kli}W_{lij} = 1$ in $U_i \cap U_j \cap U_k \cap U_l$, and changes by a coboundary if the liftings $\widetilde{V}_{ij}$ are changed. Hence W defines an element of $H^2(M, \mathbb{Z}_2)$, and this element is $\widetilde{w}_2(V)$.

Each of these dualities plays an important role in understanding conventional $\mathbf{T}^2$ compactification of the $SO(32)$ heterotic string. As we will see, they all have analogs for compactification without vector structure:

(1′) Type I compactification on $\mathbf{T}^2$ without vector structure is T-dual to a Type IIB orientifold on $\mathbf{T}^2/\mathbb{Z}_2$. However, in contrast to the usual case, there are only 8 sevenbrane pairs, and of the orientifold planes, three have sevenbrane charge -8 and one has charge $+8$.

(2′) $\mathbf{T}^2$ compactification without vector structure is equivalent to a slightly exotic version of F-theory compactification on K3 in which the K3 surface is required to have a D_8 singularity that does not produce gauge symmetry. (This result is closely related to a recent result [16] about Type IIA orientifold sixplanes, as we explain in section 4.3.)

(3′) Finally, by a heterotic string T-duality transformation, $\mathbf{T}^2$ compactification without vector structure can be mapped to an $E_8 \times E_8$ compactification on $\mathbf{T}^2$ in which the two E_8's are swapped in going around one circle in $\mathbf{T}^2$. This model was studied in [17], and argued there, in the concluding paragraph of section 2.1, to be equivalent to a Spin(32)/$\mathbb{Z}_2$ model, which is in fact the one without vector structure. This relationship has recently been discussed in more detail in [7].

Dualities (3) and (3′) in the above lists have the further implication that $\mathbf{T}^2$ compactification of the Type I superstring, with or without vector structure, has an M-theory description. Indeed, the $E_8 \times E_8$ heterotic string in ten dimensions is equivalent to M-theory on $\mathbb{R}^{10} \times \mathbf{S}^1/\mathbb{Z}_2$, so all its compactifications likewise have M-theory descriptions. The M-theory interpretation of $\mathbf{T}^2$ compactification without vector structure will not be much explored in the present paper, but will make a brief appearance in section 5.

By studying toroidal compactification without vector structure, we will get a new insight about many familiar features of orientifolds, such as the existence of different kinds of orientifold plane with orthogonal or symplectic gauge symmetry and the interplay between orthogonal and symplectic gauge symmetry on different kinds of probe, as reviewed in [18].

We begin in section 2 by studying $\mathbf{T}^2$ compactification without vector structure at the level of classical gauge theory. In the process we uncover many interesting facts that, in the stringy dualities (1′), (2′), and (3′), will appear in different ways. These include the fact that $\mathbf{T}^2$ compactification without vector structure gives a gauge group whose rank is smaller by 8 than one gets in the usual case with vector structure, and that it gives simply laced gauge groups at level two and non-simply laced groups at level one. In sections 3-5, we study the three stringy dualities listed above.

In section 3, we also briefly examine $\mathbf{T}^4$ and $\mathbf{T}^6$ compactification without vector structure.

In section 6 we study a related question suggested by this investigation. There are actually *three* supersymmetric Type IIB orientifolds on $\mathbf{T}^2/\mathbb{Z}_2$. The case in which all four orientifold planes have sevenbrane charge -8 is T-dual to Type I on $\mathbf{T}^2$ with vector structure [9, 10], and the present paper is largely devoted to showing that the case of three such planes of charge -8 and one of charge $+8$ is T-dual to the Type I string without vector structure. This leaves a third case, in which two orientifolds have charge -8, two have charge $+8$, and the number of sevenbranes is therefore zero. (This is the last model of its type, since if the net sevenbrane charge of the orientifolds is positive, one could restore neutrality only by adding anti-sevenbranes and violating supersymmetry.) In section 6, we study this model, show that it arises by dimensional reduction from a *nine*-dimensional model with orientifold planes of opposite type, and find the analogs of $(1')$ and $(2')$.

Finally, in an appendix (which can be read independently of the rest of the paper) we resolve a longstanding puzzle concerning the computation of $\mathrm{Tr}\,(-1)^F$ in four-dimensional supersymmetric gauge theories with orthogonal gauge groups.

2 Gauge theory analysis

2.1 Maximal unbroken symmetries

We will begin in this section by analyzing classical flat connections on $\mathbf{T}^2$ without vector structure.[2] In doing so, we will initially take the gauge group to be $SO(32)/\mathbb{Z}_2$ (with the $\mathbb{Z}_2$ generated by the element $-1 \in SO(32)$), and only at the end take the double cover from $SO(32)/\mathbb{Z}_2$ to $\mathrm{Spin}(32)/\mathbb{Z}_2$. Furthermore, to make the exposition somewhat clearer, we first consider $SO(4n)/\mathbb{Z}_2$ for general n, and only at the end specialize to $n = 8$.

Given a flat connection on $\mathbf{T}^2$ with values in any gauge group, one has holonomies U, V around the two factors in $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$. In our case, absence of vector structure means that U and V commute as elements of $SO(4n)/\mathbb{Z}_2$, but anticommute in $SO(4n)$:

$$UV = -VU. \tag{2.1}$$

Let us first give certain important examples of such anticommuting pairs.

$SO(4n)$ has a maximal subgroup $(Sp(1) \times Sp(n))/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ is generated by the product of $-1 \in Sp(1)$ and $-1 \in Sp(n)$. Under this decomposition, the vector representation of $SO(4n)$ decomposes as $\mathbf{4n} = \mathbf{2} \otimes \mathbf{2n}$, where $\mathbf{2}$ and $\mathbf{2n}$ are, respectively, the fundamental representations of $Sp(1)$ and $Sp(n)$. Furthermore, we have $Sp(1) \cong SU(2)$. Up to conjugation, there is a unique choice of $SU(2)$ matrices u, v with $uv = -vu$: [3]

$$\begin{aligned} u &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ v &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{2.2}$$

By taking U and V to be the $SO(4n)$ matrices $U = u \times 1$, $V = v \times 1$ (where $u \times 1$, for instance, is the product of $u \in SU(2)$ with the identity in $Sp(n)$), we get an example of an $SO(4n)$ flat connection without vector structure.

[2] For background on some parts of this discussion, see [19].

[3] The relation $vuv^{-1} = -u$ shows that u must be traceless and hence when diagonalized takes the form given in the text. Then, the fact that v anticommutes with u means that its diagonal matrix elements vanish in this basis; so up to conjugation by a diagonal matrix, v takes the claimed form.

The importance of this example is that it gives a maximal unbroken subgroup of $SO(4n)$; in fact, the given U and V obviously commute with $Sp(n)$. The unbroken symmetry group is actually somewhat bigger than this, because for an element $g \in SO(4n)$ to project to an $SO(4n)/\mathbb{Z}_2$ transformation that commutes with the projection of U and V, it is enough for g to commute *or anticommute* with U and V. This gives an extra $\mathbb{Z}_2 \times \mathbb{Z}_2$ factor in the unbroken symmetry group, since not only does $1 \times x$ (for any $x \in Sp(n)$) commute with $U = u \times 1$ and $V = v \times 1$, but $u \times x$, $v \times x$, and $uv \times x$ commute or anticommute with them. We will somewhat imprecisely refer to the subgroup of $SO(4n)$ that commutes with the Wilson lines as the unbroken symmetry group, and thus will suppress from the terminology this $\mathbb{Z}_2 \times \mathbb{Z}_2$ which is in fact always present. (The $\mathbb{Z}_2 \times \mathbb{Z}_2$ will reappear in section 3 as the basis for one explanation of why the dual torus has half the usual size.)

Presently, we will classify the possible unbroken subgroups in the above sense, and it will become clear that $Sp(n)$ is maximal in the sense that no $SO(4n)$ flat connection without vector structure has a symmetry group that contains $Sp(n)$ as a proper subgroup. However, $Sp(n)$ is not the unique maximal unbroken subgroup. Another possibility is $O(2n)$.

$O(2n)$ can arise in the following way. $SO(4n)$ has a maximal subgroup $(O(2) \times O(2n))/\mathbb{Z}_2$ (with $\mathbb{Z}_2$ generated by the product of $-1 \in O(2)$ and $-1 \in O(2n)$) under which the vector representation of $SO(4n)$ decomposes as $\mathbf{4n} = \mathbf{2} \otimes \mathbf{2n}$, with the two factors now the vector representations of $O(2)$ and $O(2n)$. The relation $uv = -vu$ can be obeyed by the $O(2)$ matrices

$$\begin{aligned} u &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ v &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{2.3}$$

By setting $U = u \times 1$, $V = v \times 1$, we get a flat $SO(4n)$ bundle without vector structure with unbroken gauge group $O(2n)$. $O(2n)$ cannot be embedded in $Sp(n)$, and is a new example of a maximal symmetry group.

The $Sp(n)$ and $O(2n)$ constructions differ from each other in the following way. Up to a gauge transformation, there is only one flat $SO(4n)$ bundle without vector structure with unbroken $Sp(n)$. This is so because the structure group of such a bundle would reduce to $SU(2)$ (the commutant of $Sp(n)$), but anticommuting matrices $u, v \in SU(2)$ are unique up to conjugation (as was proved in the footnote accompanying (2.2)). However, in $O(2)$ it is not true that the choice of group elements u, v with $uv = -vu$ is unique up to gauge transformation. With the specific choice in (2.3), we have $\det u = -1$, $\det v = +1$. These conditions are not invariant under replacing (u, v) by (v, u) or (uv, v) (operations that do preserve $uv = -vu$), so we get at least three gauge-inequivalent flat bundles with unbroken $O(2n)$. Up to conjugation there are precisely these three possibilities; this may be proved as follows. Consider any flat $O(2)$ connection on $\mathbf{T}^2$ without vector structure. Let Γ be a two-dimensional lattice with $\mathbf{T}^2 = \mathbb{R}^2/\Gamma$. Define a homomorphism $\phi : \Gamma \to \mathbb{Z}_2$ (with $\mathbb{Z}_2$ regarded as the multiplicative group $\{\pm 1\}$) by mapping each $\gamma \in \Gamma$ to the determinant of the holonomy of the given $O(2)$ connection around the one-cycle in $\mathbf{T}^2$ that corresponds to γ. The homomorphism ϕ must be non-trivial; if it were trivial, the flat $O(2)$ connection would actually be an $SO(2)$ connection, but as $SO(2)$ is abelian this would make the relation $uv = -vu$ impossible. There are three non-zero possibilities for ϕ; the three possibilities are that the restrictions of ϕ to a basis of the lattice can be $(-1, 1)$, $(1, -1)$, or $(-1, -1)$. An $O(2)$ flat connection without vector structure is uniquely determined up to gauge transformation by its associated ϕ. For example, in the $(-1, 1)$ case, the holonomy u around the first circle has determinant -1, and so u is conjugate to the matrix given in (2.3). With this choice of u, the fact that the holonomy v around the second circle is of determinant 1 and anticommutes with u leads (up to conjugation by u) to the choice in (2.3).

The mapping class group $SL(2,\mathbb{Z})$ of $\mathbf{T}^2$ acts on the moduli space of flat connections without vector structure. The uniqueness of the point with unbroken $Sp(n)$ implies that it is $SL(2,\mathbb{Z})$-invariant. But the three points with unbroken $O(2n)$ are permuted by $SL(2,\mathbb{Z})$ like the ϕ's, that is, like the three real line bundles of order two on $\mathbf{T}^2$ (or equivalently, like the half-lattice points of Γ).

Maximal symmetry groups other than those that we have seen so far can be obtained by starting with $SO(4k) \times SO(4(n-k)) \subset SO(4n)$, and making one of the two constructions that we have so far seen in $SO(4k)$ and the other in $SO(4(n-k))$. In this way we get flat bundles with unbroken symmetry $Sp(k) \times O(2(n-k))$ for any k. As we will prove below, this is the complete list of possible maximal unbroken symmetry groups. However, if we define a locally maximal unbroken symmetry to be the symmetry of a flat connection that has the property that under any small perturbation the gauge symmetry would be reduced, then there are other locally maximal examples. They can be built starting with $\prod_{i=1}^4 SO(4k_i) \subset SO(4n)$, with $\sum_{i=1}^4 k_i = n$, and making the Sp construction in the first factor and the three SO constructions in the other three factors. It will soon be clear that all of these statements have an intuitive explanation in terms of sevenbranes.

2.2 Systematic classification

It follows from standard theorems about flat connections on Riemann surfaces that the moduli space of flat connections on $\mathbf{T}^2$ without vector structure is connected,[4] so it must be possible to continuously interpolate between the examples that we have given of flat connections without vector structure. We will now describe this interpolation explicitly,

[4]The equivalence between flat unitary connections and stable holomorphic bundles on a Riemann surface means that a flat connection A can be reconstructed up to gauge transformation from its $\overline{\partial}$ operator $\overline{\partial}_A$. Given two flat connections A and A' on the same bundle, one can continuously interpolate between the $\overline{\partial}$ operators by looking at the family $t\overline{\partial}_A + (1-t)\overline{\partial}_{A'}$, with t varying from 0 to 1; this interpolation between the $\overline{\partial}$ operators induces an interpolation between the flat connections. It is possible to make this interpolation because in complex dimension one there is no integrability condition for $\overline{\partial}$ operators. The absence of obstructions to the deformation of a $\overline{\partial}$ operator in complex dimension one also implies that the moduli spaces of flat unitary connections are irreducible (they do not have different components meeting on a submanifold).

and in the process will see that the possible maximal symmetry groups of a flat connection without vector structure are those listed in the last paragraph.

Suppose that we start at a flat connection (on an $SO(4n)/\mathbb{Z}_2$ bundle without vector structure) with unbroken $Sp(n)$, and consider a small perturbation of the flat connection. Any small perturbation can be made by introducing Wilson lines of the unbroken subgroup, in this case $Sp(n)$. So let u', v' be any *commuting* elements of $Sp(n)$, and let $U = u \times u'$, $V = v \times v'$, where u, v are anticommuting elements of $SU(2)$. This gives a family of flat connections without vector structure.

Since u' and v' commute, they can be simultaneously conjugated to a maximal torus of $Sp(n)$. Such a torus is contained in a subgroup $Sp(1)^n$ of $Sp(n)$. As elements of $Sp(1)^n$, we can write $u' = \prod_{i=1}^n u_i$, $v' = \prod_{i=1}^n v_i$, where u_i and v_i are the $Sp(1)$ or $SU(2)$ elements

$$\begin{aligned} u_i &= \begin{pmatrix} e^{i\theta_i} & 0 \\ 0 & e^{-i\theta_i} \end{pmatrix} \\ v_i &= \begin{pmatrix} e^{i\psi_i} & 0 \\ 0 & e^{-i\psi_i} \end{pmatrix}. \end{aligned} \tag{2.4}$$

It will sometimes be convenient to combine θ_i and ψ_i in the pair $a_i = (\theta_i, \psi_i)$. The Weyl group of $Sp(n)$ acts by permutations of the $Sp(1)$ factors and by Weyl transformations of the individual factors. The permutations act by permutations of the i index of a_i, and the $Sp(1)^n$ Weyl transformations act by $a_i \to \pm a_i$, with independent choices of sign for each i. Without this sign ambiguity, we would interpret each a_i as a point on a dual torus $\widetilde{\mathbf{T}}^2$ (which can be interpreted as the moduli space of $U(1)$ flat connections on $\mathbf{T}^2$). Modulo the sign ambiguity, each a_i determines a point on an orientifold $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. The complete set $a_1, a_2, \ldots, a_n$ modulo permutations is a collection of n unordered points on $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. This is in fact the moduli space of flat $Sp(n)$ connections on $\mathbf{T}^2$.

However, the map from the $Sp(n)$ moduli space to the $SO(4n)/\mathbb{Z}_2$ space, given by $u', v' \to U = u \times u'$, $V = v \times v'$, is not one-to-one. This map is surjective and is locally an isomorphism at a generic point in the moduli spaces,[5] but is a many-to-one map because there are discrete identifications in $SO(4n)/\mathbb{Z}_2$ that would not be present in $Sp(n)$.

These discrete identifications are independent shifts of θ_i and ψ_j by π, generating in all a group Θ with 2^{2n} elements. To see these discrete identifications, note that $SO(4n)$ contains a subgroup $(SO(4))^n$. We identify $SO(4)$ with $(SU(2) \times SU(2))/\mathbb{Z}_2$, and refer to the factors as the "first" and "second" $SU(2)$'s. We align the $(SO(4))^n$ subgroup with the $Sp(1) \times Sp(n)$ that was used earlier in such a way that $Sp(1)$ is a diagonal subgroup of the product of the n "first" $SU(2)$'s, and $Sp(1)^n \subset Sp(n)$ is the product of the n "second" $SU(2)$'s. In this setup, the group element $U = u \times u'$, with $u' = \prod_i u_i$, is the

[5] Upon picking a complex structure on $\mathbf{T}^2$ and identifying the moduli spaces of flat connections with moduli spaces of stable bundles, the map between $Sp(n)$ and $SO(4n)/\mathbb{Z}_2$ moduli spaces is holomorphic. Since this map is also an isomorphism near $u' = v' = 1$, and the target space is irreducible according to the previous footnote, the map is surjective and in fact the $Sp(n)$ moduli space is a finite cover of the $SO(4n)/\mathbb{Z}_2$ moduli space. The covering group is determined by the discrete identifications that are explained momentarily.

JHEP02(1998)006

element $U = \prod_i (u \times u_i)$ of $(SO(4))^n$, and likewise we can write $V = \prod_i (v \times v_i)$. Now, let W_k be the element of $SO(4)^n$ whose k^{th} factor is $v \times 1$, while the other factors are 1. Then conjugation by W_k shifts θ_i by $\pi\delta_{ik}$ while leaving ψ_j invariant. Similarly, if X_k is the element of $SO(4)^n$ whose k^{th} factor is $u \times 1$ while the others are 1, then conjugation by X_k shifts ψ_j by $\pi\delta_{jk}$ and leaves θ_i unchanged. These transformations thus generate the full group Θ of independent π shifts of all components of the a_i. (A more intuitive explanation of the origin of the group Θ is that it comes by combining the Weyl groups of the various $Sp(n)$ and $O(2n)$ maximal unbroken symmetry groups.)

The fundamental domain of θ_i, ψ_j, subject to these symmetries can be taken to be

$$
\begin{aligned}
-\frac{1}{2}\pi \leq \theta_i \leq \frac{1}{2}\pi \\
-\frac{1}{2}\pi \leq \psi_j \leq \frac{1}{2}\pi.
\end{aligned}
\tag{2.5}
$$

Since the endpoints at $\pm\pi/2$ are identified, the θ_i and ψ_j are still angular variables, but with half the usual period. Hence, each a_i defines a point in a torus, which we will call $\widetilde{\mathbf{T}}^2$, but (despite our unchanged notation) this is *not* the usual dual torus; it is twice as small in each direction, and its area is one-fourth the area of the usual dual torus. We still have to divide by the Weyl group of $Sp(n)$, but this can be done just as in the case of ordinary $Sp(n)$ flat connections that was treated earlier. The sign changes $a_i \to -a_i$ (with independent signs for each i) mean that each a_i defines a point on an orientifold $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. Including the i index and dividing by permutations, the moduli space of $SO(4n)/\mathbb{Z}_2$ connections without vector structure is the configuration space of n unordered points on $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. Those points will be interpreted as positions of sevenbranes, roughly as in the more familiar case [9, 10] with vector structure.

Finally, let us identify the unbroken gauge symmetry group for each set of sevenbrane positions. Generically, the unbroken symmetry group is a subgroup of $Sp(n)$. When it is such a subgroup, it is easy to determine which one. For completely generic a_i, the unbroken subgroup of the gauge group is a maximal torus $U(1)^n$ of $Sp(n)$. If k of the a_i coincide at a generic point, the associated $U(1)^k$ is enhanced to a $U(k)$ subgoup of $Sp(n)$, just as for standard orientifolds. If k of the a_i coincide at $a_i = 0$, the associated $U(1)^k$ is enhanced to an $Sp(k)$ subgroup.

The interesting phenomenon that remains is that for certain special (and non-zero) values of the a_i, there is an unbroken symmetry group that is *not* a subgroup of $Sp(n)$. This happens, in fact, precisely when some of the a_i are at orientifold fixed points in $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ that are *not* at $a_i = 0$. To analyze this, we can proceed as follows. The part of the Lie algebra of $SO(4n)$ that is not in the Lie algebra of $Sp(1) \times Sp(n)$ transforms in the representation $R = \mathbf{3} \otimes \mathbf{A}$, where $\mathbf{3}$ is the adjoint representation of $SU(2)$ and $\mathbf{A}$ is the traceless second rank antisymmetric tensor of $Sp(n)$. Enhanced gauge symmetry not contained in $Sp(n)$ will arise if and only if there are non-zero vectors in R that are invariant under the holonomies U and V. Let us determine the condition for this.

The $\mathbf{3}$ of $SU(2)$ can be identified as the vector representation of $SO(3)$, and if we think about it this way, then u and v (which commute in $SO(3) = SU(2)/\mathbb{Z}_2$) can be identified with diagonal matrices

$$
\begin{aligned}
u &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
v &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\end{aligned}
\tag{2.6}
$$

Since these matrices are diagonal in this basis, the subspace of R that is invariant under U and V is a direct sum of joint eigenspaces of u and v. For instance, we can look for vectors ψ with $u\psi = -\psi$, $v\psi = \psi$.

For such a ψ to be invariant under U and V, it must obey also $u'\psi = -\psi$, $v'\psi = \psi$. Let us analyze these conditions. Let $\mathbf{2}_i$ be the two-dimensional representation of the i^{th} copy of $Sp(1)$ in $(Sp(1))^n \subset Sp(n)$. Then the $Sp(n)$ representation $\mathbf{A}$ is the sum $\oplus_{1\leq i<j\leq n}\mathbf{2}_i \otimes \mathbf{2}_j$ plus "diagonal" terms on which $Sp(1)^n$ acts trivially; the diagonal terms cannot contribute vectors with $u'\psi = -\psi$. The action of U and V is block-diagonal in the decomposition of $\mathbf{A}$ in terms of $\oplus_{1\leq i<j\leq n}\mathbf{2}_i \otimes \mathbf{2}_j$, so ψ can be taken to be a sum of vectors in $\mathbf{2}_i \otimes \mathbf{2}_j$. The existence of a non-zero vector $\psi \in \mathbf{2}_i \otimes \mathbf{2}_j$ with $u'\psi = -\psi$, $v'\psi = \psi$ implies that (modulo sign changes of the θ's generated by Weyl transformations)

$$
\begin{aligned}
\theta_i = \theta_j &= \pi/2 \\
\psi_i &= -\psi_j.
\end{aligned}
\tag{2.7}
$$

Such a configuration can actually be mapped to $a_i = a_j$, which gives enhanced gauge symmetry in $Sp(n)$, by a transformation in Θ (namely $\theta_j \to \theta_j + \pi$) plus a Weyl transformation (changing the sign of a_j). Hence, for generic ψ_i and ψ_j, this mechanism gives nothing essentially new. But if we actually have $\psi_i = \psi_j = 0$, then in addition to (2.7), we have $a_i = a_j$, giving enhanced gauge symmetry *inside* $Sp(n)$ as well as the enhanced gauge symmetry that appears because (2.7) is obeyed. It is impossible by a Θ plus Weyl transformation to map everything into $Sp(n)$, so from this configuration we get something essentially new.

The relevant case is thus

$$
a_i = a_j = (\pi/2, 0). \tag{2.8}
$$

Similarly, symmetry enhancement in other eigenspaces of u, v that cannot be conjugated into $Sp(n)$ is associated with some of the a's being located at one of the other nonzero orientifold fixed points, namely $(0, \pi/2)$ or $(\pi/2, \pi/2)$.

If precisely k of the a_i lie at $(\pi/2, 0)$ (or one of the other non-zero orientifold fixed points), then the unbroken symmetry subgroup of $Sp(n)$ associated with the a_i is $U(k)$. In addition, one gets out of each of the $k(k-1)/2$ relevant $\mathbf{2}_i \otimes \mathbf{2}_j$'s, precisely two unbroken generators that are *not* in $Sp(n)$. By (for example) analyzing the action of a maximal torus of $U(k)$, it can be seen that these transform as the second rank antisymmetric tensor of $U(k)$ plus its dual, so the Lie algebra of the unbroken symmetry group is actually that of $SO(2k)$. With more care, one can see that the unbroken symmetry group is $O(2k)$.

If then we set $k = n$, we get, for each of the non-zero orientifold fixed points, a flat connection without vector structure with unbroken $O(2n)$. These of course were written down by hand at the beginning of the present section. More generally, if one distributes all a_i among the four orientifold fixed points in an arbitrary fashion, one gets the locally maximal unbroken symmetry groups mentioned at the end of the section 2.1.

2.3 Comparison with standard orientifold

Now we want to set $n = 8$, so that $SO(4n)$ becomes the $SO(32)$ of the heterotic or Type I superstring, and compare to the standard analysis of orientifolds.

The description of the moduli space that we have found is quite similar to the one that is associated with the standard orientifold with vector structure [9, 10]. The moduli space of $\mathbf{T}^2$ compactification without vector structure can be described in terms of branes on an orientifold $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$, but there are some important differences from the usual case:

(1) The orientifold is half as large in each direction as in the usual case, and has therefore only one-fourth the usual area.

(2) There are only eight sevenbranes on the orientifold (or eight pairs of sevenbranes on the covering space $\widetilde{\mathbf{T}}^2$), which is just half the number on the standard orientifold.

(3) The four orientifold planes, which are derived from the fixed points of the $\mathbb{Z}_2$ action on $\widetilde{\mathbf{T}}^2$, are of two different kinds. If k sevenbranes meet the orientifold plane at $a = 0$, then $Sp(k)$ gauge symmetry is produced, while k sevenbranes at one of the other three orientifold planes produce $O(2k)$ gauge symmetry.

(4) The rank of the gauge group is reduced by eight compared to what one has in standard $\mathbf{T}^2$ compactification. While standard $\mathbf{T}^2$ compactification gives only simply-laced symmetry groups at level one, here we get the non-simply-laced $Sp(k)$ groups at level one and simply-laced $SU(n)$ or $SO(2n)$ at level two.

A few additional words of explanation concerning these points should suffice. The statement that the orientifold is half the usual size (one-fourth the usual area) was obtained in eqn. (2.5). The fact that there are only eight sevenbranes, which is clear from the rank of the unbroken gauge group, is correlated with the fact that there are orientifold planes of both kinds. Unoriented strings can have Sp or SO Chan-Paton factors (for a review see section 1.3 of [20]; see also [21] concerning the restriction to classical groups), and related to this, their T-duals can have two kinds of orientifold plane, which produce respectively Sp and SO gauge symmetry (for reviews see [18]). We will call them $\mathcal{O}^-$ and $\mathcal{O}^+$. They produce tadpoles of opposite sign; $\mathcal{O}^+$ and $\mathcal{O}^-$ sevenplanes have sevenbrane charges -8 and $+8$ respectively. The $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ orientifold that is T-dual to standard $\mathbf{T}^2$ compactification of Type I has four orientifold planes of type $\mathcal{O}^+$. In this case, the orientifolds carry a total charge of -32, so one requires 16 pairs of sevenbranes. In the present case, the appearance of Sp symmetry at one sevenbrane and SO at the other three shows that we have three $\mathcal{O}^+$ planes and one $\mathcal{O}^-$ plane, with a net sevenbrane charge of -16.

We therefore should expect eight pairs of sevenbranes instead of 16, and the rank of the unbroken subgroup of $SO(32)$ should be 8. This agrees with what we have found from classical gauge theory (since for $SO(4n)/\mathbb{Z}_2$ the unbroken gauge group has rank n).

Finally, thinking in terms of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string, and considering only the gauge symmetries that arise for generic radius (other cases were studied in [17]), the $Sp(k)$ gauge symmetry in $\mathbf{T}^2$ compactification without vector structure is at level one, since the embedding $Sp(k) \subset Sp(n) \subset Sp(1) \times Sp(n) \subset SO(4n)$ is a level one embedding of $Sp(k)$ in $SO(4n)$. But the $U(k)$ and $SO(2k)$ gauge symmetries are at level two, since their embeddings in $SO(4n)$ by $U(k) \subset SO(2k) \subset SO(2n) \subset O(2) \times O(2n) \subset SO(4n)$ are at level two.

Throughout this section, we have considered $SO(32)/\mathbb{Z}_2$ flat connections without vector structure, while in the heterotic or Type I superstring one really wants $\mathrm{Spin}(32)/\mathbb{Z}_2$. The difference only arises if one considers particles in spin representations of $\mathrm{Spin}(32)/\mathbb{Z}_2$. These are not seen in classical gauge theory, but arise in the Type I description from Dirichlet one-branes wrapped on a cycle in $\mathbf{T}^2$, which become one-branes on $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. In a more precise description that takes account of these states, we must distinguish two configurations of sevenbranes that differ by a motion of any one sevenbrane around a cycle in the orientifold. The same remark applies for the more familiar [9, 10] orientifold that is related to compactification with vector structure.

In this section, we have analyzed only classical gauge theory and not string theory. This has sufficed to see what kind of orientifold must be T-dual to the Type I superstring on $\mathbf{T}^2$ without vector structure. In the next section, we will actually analyze the T-duality and show how the expected answer arises.

3 T-duality

3.1 The closed string sector

Our goal in the present section is to apply T-duality to Type I compactification on $\mathbf{T}^2$ without vector structure. The aim is to explain and recover via T-duality the features that were obtained by an analysis of classical gauge theory in section 2 and summarized in section 2.3. We especially want to explain why the dual torus is of half the usual size (one quarter the usual area) and why there are three orientifold planes of type $\mathcal{O}^+$ and one of type $\mathcal{O}^-$.

We will begin with the closed strings, which are of course unoriented, and then move on to the open strings. At first sight we face a quandary. The closed strings do not appear to "know" whether there is vector structure or not, since gauge fields appear only in the open string sector. So how can application of T-duality to the closed strings produce a torus of half the size in case there is no vector structure?

What saves the day is that there is in fact a subtle correlation between the presence or absence of vector structure and the couplings in the closed string sector. We recall that for Type IIB superstrings, there is a two-form that comes from the NS sector, and another from the Ramond sector. In compactification on $\mathbf{T}^2$, the NS two-form gives rise to a world-sheet theta angle whose effects can be seen in string perturbation theory. (The Ramond two-form gives a second theta-like angle with nonperturbative effects.) The orientifold projection to Type I removes the NS two-form; the Ramond two-form survives. As for the world-sheet theta angle, it is odd under reversal of world-sheet orientation, so in Type I it is severely restricted; it must be 0 or π.

At first sight, it seems therefore that there are four Type I models on $\mathbf{T}^2$ that we could potentially consider. One may have or not have vector structure; and the theta angle may be 0 or π. However [22], these choices are correlated; the theta angle is π if and only if there is no vector structure. So to study the theory without vector structure, we must set $\theta = \pi$. In this way, the closed string sector does "know" that there is no vector structure.

Now, let us recall the structure of the T-duality group of $\mathbf{T}^2$. It is $SO(2,2;\mathbb{Z}) \cong SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$. The first $SL(2,\mathbb{Z})$ acts on the complex structure of $\mathbf{T}^2$ and will play practically no role in what follows. The second acts in the customary way on

$$\tau = iA + \frac{\theta}{2\pi} \tag{3.1}$$

where A is the area of $\mathbf{T}^2$ and θ is the theta angle.

What $SL(2,\mathbb{Z})$ transformation do we want to make? Application of any $SL(2,\mathbb{Z})$ transformation at all will give a correct result. But our goal in applying T-duality will be to determine the behavior of the model in the limit of small A, by finding an $SL(2,\mathbb{Z})$ transformation that will map us back to large A. At $\theta = 0$, $\tau \to -1/\tau$ reduces to $A \to 1/A$, and so maps small A to large A. But at $\theta = \pi$, $\tau \to -1/\tau$ maps small A to small A and is not helpful.

A transformation that is more helpful at $\theta = \pi$ is

$$\tau \to \frac{\tau - 1}{2\tau - 1}. \tag{3.2}$$

A short calculation reveals that at $\theta = \pi$, this transformation reduces to

$$A \to \frac{1}{4A} \tag{3.3}$$

(with no change in θ), so it has the desired effect of mapping small area back to large area. We also see that, as anticipated via classical gauge theory in section 2, the area of the dual torus is four times smaller than it is for the usual T-duality of $\mathbf{T}^2$. Since the transformation (3.2) acts trivially on the complex structure of the torus, the four-fold reduction in the area is achieved by reducing all lengths by an extra factor of two compared to the standard T-duality at $\theta = 0$.

So we have obtained a "stringy" explanation of a result in section 2, but why the particular $SL(2,\mathbb{Z})$ transformation indicated in (3.2) is the right one at $\theta = \pi$ begs for a more intuitive explanation.

Intuitive explanation of $R \to 1/(2R)$

For this, we need to look at the T-duality more explicitly. (See [23] for background and further detail. The rest of this paper does not depend on the following intuitive explanation.) We consider a rectangular $\mathbf{T}^2$ with radii R_1, R_2. The usual T-duality, at $\theta = 0$, is $(R_1, R_2) \to (1/R_1, 1/R_2)$. It is convenient to combine this with a $\pi/2$ rotation, to give $(R_1, R_2) \to (1/R_2, 1/R_1) = A^{-1}(R_1, R_2)$. This makes it explicit that this transformation inverts the area without changing the complex structure.

T-duality acts on a four-dimensional lattice consisting of momenta p_x, p_y along the two directions in $\mathbf{T}^2$ and corresponding windings m_x, m_y. (We measure momenta and winding in units such that the p's and m's are integral at $\theta = 0$.) For our purposes, we can reduce to a two-dimensional lattice, as follows. An $R \to 1/R$ transformation is $p_x \leftrightarrow m_y$ and $p_y \leftrightarrow -m_x$. (Of course, p_x is exchanged with m_y instead of m_x since we combine the $R \to 1/R$ transformation with a $\pi/2$ rotation; for the same reason p_y maps to $-m_x$.) This transformation, together with $\theta \to \theta + 2\pi$, generates the $SL(2,\mathbb{Z})$ that acts on A and θ.

At non-zero θ, the p's are shifted from their conventional integral values. In fact, p_x is shifted from an integer by $m_y(\theta/2\pi)$, and p_y by $-m_x(\theta/2\pi)$. The transformation $\theta \to \theta + 2\pi$ is hence $p_x \to p_x + m_y$, $p_y \to p_y - m_x$. In particular, the lattice generated by p_x and m_y (and likewise the lattice generated by p_y and m_x) is mapped to itself by $\theta \to \theta + 2\pi$ as well as by $R \to 1/R$, so it is mapped to itself by the $SL(2,\mathbb{Z})$ that acts on A and θ. We can focus just on the p_x, m_y lattice.

Now let us see what happens at $\theta = \pi$. The p_x, m_y lattice at this value of θ is generated by the vectors $f = (1,0)$ and $e = (-1/2, 1)$. Clearly, such a lattice cannot have a symmetry that exchanges p_x with m_y, since such a symmetry would map $(-1/2, 1)$, which is in the lattice, to $(1, -1/2)$, which is not. However, let $h = (0,2)$. The lattice is generated by f, h, and $\frac{1}{2}(f+h)$, a set of vectors that is symmetric under $f \leftrightarrow h$, so it has a symmetry W that acts by $f \to h$, $h \to -f$. (The minus sign ensures that the determinant is 1, so that W is in $SL(2,\mathbb{Z})$.) Thus W maps p_x to $m_y/2$. The factor of 2 is what we are looking for. We start with a radius R in the x direction, so the $(1,0)$ state has energy $1/R$. If the W transformation produces a dual circle with radius $\widetilde{R}$, then the $(0,2)$ state after duality has energy $2\widetilde{R}$. So $1/R = 2\widetilde{R}$, that is $\widetilde{R} = 1/(2R)$. So the natural symmetry at $\theta = \pi$ is $R \to 1/(2R)$, as we have already discovered in several different ways.

The transformation W, which maps $f \to h$ and $h \to -f$, can be written in terms of the original basis e, f as $f \to f + 2e$, $e \to -f - e$. This corresponds in the basis (f, e) to the $SL(2,\mathbb{Z})$ matrix

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}. \tag{3.4}$$

On the upper half plane, this acts by $\tau \to (\tau - 1)/(2\tau - 1)$, which is the transformation whose origin we wished to explain intuitively.

Nature of the orientifold planes

So far, our discussion applies most naturally to *oriented* closed strings at $\theta = \pi$. Now, we must adapt the discussion to *unoriented* closed strings, relevant to Type I.

For the same reasons as in [9, 10], projecting the closed strings on the original torus $\mathbf{T}^2$ onto states invariant under reversal of orientation is equivalent in the T-dual description to replacing the dual torus $\widetilde{\mathbf{T}}^2$ by an orientifold $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. We recall that this is argued as follows. If p_L and p_R are the left and right-moving momenta of a closed string, then $R \to 1/R$ (or $R \to 1/(2R)$) acts by $T : p_R \to p_R, p_L \to -p_L$. On the other hand, world-sheet orientation reversal is $\Omega : p_L \to p_R, p_R \to p_L$. Ω transforms under T-duality to $T\Omega T^{-1}$, which acts by $p_L \to -p_L, p_R \to -p_R$. But, since it acts in the same way on p_L and p_R, this is a geometrical transformation. In fact, it acts on the dual torus by "multiplication by -1"; projecting onto states invariant under it replaces $\widetilde{\mathbf{T}}^2$ by $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$.

There are four fixed points in the $\mathbb{Z}_2$ action on $\widetilde{\mathbf{T}}^2$, and each becomes an orientifold plane. Now, however, we wish to explain in terms of closed string T-duality another important result of section 2, which is that in the case without vector structure, the orientifolds have a net sevenbrane charge of -16, compared to -32 for the case with vector structure. We will do this by carefully comparing the worldsheet path integrals for the case that the worldsheet topology is $\mathbf{S}^2$ or $\mathbf{RP}^2$.

Let A be the area of the original $\mathbf{T}^2$, A' the area of the dual $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ obtained by $R \to 1/R$ at $\theta = 0$, and A'' the area of the dual $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ obtained by $R \to 1/(2R)$ at $\theta = \pi$. (Thus, $A'' = A'/4$, a fact that will be used presently.) Let also g be the string coupling constant in the original description, g' the string coupling constant in the T-dual at $\theta = 0$, and g'' the string coupling constant in the T-dual at $\theta = \pi$.

Before T-duality, the world-sheet path integral for a worldsheet Σ that is topologically $\mathbf{S}^2$ or $\mathbf{RP}^2$ is independent of θ, since a worldsheet of the given topology cannot "wrap" around $\mathbf{T}^2$. We want to impose the condition that also after T-duality, the path integral is the same whether we are at $\theta = 0$ or $\theta = \pi$.

First we consider the case that the worldsheet is $\mathbf{S}^2$. The dependence on the coupling and area is extremely simple. Before T-duality, as there are no wrapping modes the worldsheet path integral is simply proportional to A; and of course in genus zero one has a factor of g^{-2}. Likewise after T-duality there are still no wrapping modes in genus zero, so the partition function is proportional to $A'/(g')^2$ or $A''/(g'')^2$. All other factors are the same whether θ is 0 or π. So equality of the partition functions after T-duality gives

$$\frac{A'}{(g')^2} = \frac{A''}{(g'')^2}. \tag{3.5}$$

Since $A'' = A'/4$, we get

$$g'' = \frac{g'}{2}. \tag{3.6}$$

Now, let us compare the $\mathbf{RP}^2$ partition functions, after T-duality, at $\theta = 0$ and $\theta = \pi$. For $\mathbf{RP}^2$, the partition function has only one power of inverse string coupling, instead of two. Also, there is no factor of area in the partition function of the orientifold, because the "center of mass" of the worldsheet is always mapped to one of the orientifold planes. The partition function for $\mathbf{RP}^2$ therefore contains a sum over orientifold planes, but it is not simply proportional to the number of such planes. There are two kinds of orientifold planes, called $\mathcal{O}^+$ and $\mathcal{O}^-$ in section 2, which contribute $\mathbf{RP}^2$ partition functions of equal and opposite signs. If therefore n_+ and n_- are the number of $\mathcal{O}^+$ and $\mathcal{O}^-$ planes – so that

$$n_+ + n_- = 4 \tag{3.7}$$

regardless of the value of θ – then the worldsheet path integral for $\mathbf{RP}^2$ is proportional to $\Delta = n_+ - n_-$. Letting Δ', Δ'' be the values of Δ at $\theta = 0$ and $\theta = \pi$, respectively, we have therefore

$$\frac{\Delta'}{g'} = \frac{\Delta''}{g''}. \tag{3.8}$$

In view of (3.6), this gives $\Delta'' = \frac{1}{2}\Delta'$. Since the standard orientifold at $\theta = 0$ has $n_+ = 4$, $n_- = 0$, and $\Delta' = 4$, it follows that the $\theta = \pi$ orientifold has $\Delta'' = 2$, and hence $n_+ = 3$ and $n_- = 1$. The values of n_+ and n_- are of course in agreement with what we found in section 2 using classical gauge theory.

3.2 T-duality for open strings

So far, we have only considered T-duality in the closed string sector. Our next task is to analyze T-duality for open strings. First we consider ordinary open strings of the underlying Type I model – sometimes called strings that end on ninebranes. Then, we will analyze T-duality for Dirichlet onebrane probes. The case of fivebranes will be postponed to section 4, and will serve as the stepping stone to an F-theory description.

We examine the ordinary open strings first in the vacuum with unbroken $Sp(8)$. The Chan-Paton factors at the end of an open string transform in the $\mathbf{32}$ of $SO(32)$, which is the $\mathbf{2} \otimes \mathbf{16}$ of $(Sp(1) \times Sp(8))/\mathbb{Z}_2 \subset SO(32)$. An actual open string has a charge in this representation at each end, so the open string itself transforms as $(\mathbf{2} \otimes \mathbf{16}) \otimes (\mathbf{2} \otimes \mathbf{16}) \cong (\mathbf{2} \otimes \mathbf{2}) \otimes (\mathbf{16} \otimes \mathbf{16})$, which in particular (unlike the $\mathbf{32}$ itself) is a representation of the group $\mathrm{Spin}(32)/\mathbb{Z}_2$.

The $\mathbf{2} \otimes \mathbf{2}$ of $Sp(1) = SU(2)$ can be decomposed in the usual way as $\mathbf{1} \oplus \mathbf{3}$. Since this is a representation of $SO(3) = SU(2)/\mathbb{Z}_2$ (with no need for the double covering to $SU(2)$), the $SU(2)$ Wilson lines u, v commute in this representation, and so can be simultaneously diagonalized. In fact, in the $\mathbf{3}$ this diagonalization was already made in eqn. (2.6), while in the $\mathbf{1}$, u and v of course have the common eigenvalue 1. Combining these results, the $\mathbf{1} \oplus \mathbf{3}$ has the property that the eigenvalues of the pair (u, v) run over all possible pairs $(\pm 1, \pm 1)$, with each of the four combinations of signs appearing precisely once. An eigenvalue -1 for u or v shifts the possible momenta of a particle or string by half a unit. The fact that all pairs $(\pm 1, \pm 1)$ appear with unit multiplicity means that all half-integral shifts appear with the same multiplicity as the unshifted momenta. Hence, the momenta of the open strings take values not in the usual momentum lattice Λ but in a rescaled lattice $\frac{1}{2}\Lambda$.

After T-duality, the momentum lattice of the open strings is reinterpreted as a lattice of windings for open strings that now obey Dirichlet boundary conditions. To obtain a lattice Λ of windings, one usually has open strings on a dual torus $\widetilde{\mathbf{T}}^2 = \mathbb{R}^2/\Lambda$. In the present case, to get the desired rescaled winding lattice $\frac{1}{2}\Lambda$, we must define $\widetilde{\mathbf{T}}^2 = \mathbb{R}^2/(\frac{1}{2}\Lambda)$. So we recover the result that we have by now obtained in several other ways: in compactification without vector structure, the dual torus has one-half the usual size.

Since the open strings are unoriented, a winding $\lambda \in \frac{1}{2}\Lambda$ must be identified with $-\lambda$. (Also, a certain projection must be made on the states with $\lambda = 0$.) In the usual way [9, 10], this means that $\widetilde{\mathbf{T}}^2$ must be replaced by the orientifold $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. Also, the vacuum with unbroken $Sp(8)$ has precisely eight sevenbrane pairs at the $\mathcal{O}^-$ fixed point in $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. We of course get the other maximal unbroken symmetry, $O(16)$, if all sevenbranes are placed at an $\mathcal{O}^+$ fixed point.

It is perhaps easier to understand this construction if we compare to the usual case with vector structure. In that case, one has 16 sevenbrane pairs. If one wants the windings to take values in $\frac{1}{2}\Lambda$ and not Λ, one must divide the sevenbrane pairs into four groups (of four each), and place one group at each orientifold fixed point. In this case, the unbroken gauge group is $SO(8)^4$, with one factor from each of the four groups of sevenbranes. In particular, the unbroken group is not simple. The other way to get a winding lattice $\frac{1}{2}\Lambda$ is the one followed in the theory without vector structure: take the dual torus to be half as big. In this case, one can get the simple unbroken gauge group $Sp(8)$ or $O(16)$.

T-duality for one-brane probes

Our next target will be to analyze T-duality for Dirichlet onebrane probes of this system.

We begin with Type I on $\mathbb{R}^8 \times \mathbf{T}^2$ without vector structure. Consider a Dirichlet onebrane whose world-volume is localized at a (fluctuating) point on $\mathbf{T}^2$, and occupies a two-dimensional subspace $\mathbb{R}^2$ of $\mathbb{R}^8$. Such a onebrane behaves as a "solitonic heterotic string," and is transformed to an elementary heterotic string under heterotic/Type I duality. Under T-duality to the $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ orientifold, it becomes a threebrane whose world-volume is $\mathbb{R}^2 \times \widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ (with a further subtlety that will appear).

The point that we wish to investigate is the following. Consider the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string on $\mathbf{T}^2$ without vector structure. For certain choices of Wilson line, as analyzed in detail in section 2, one gets an unbroken $U(k)$ gauge symmetry. As noted in section 2.3, this is a level two gauge symmetry, meaning that on the worldvolume of an extended heterotic string transverse to the $\mathbf{T}^2$, there will be a level two $U(k)$ current algebra.

After heterotic/Type I duality, it follows that an extended Dirichlet onebrane transverse to the $\mathbf{T}^2$ will have such a level two current algebra. T-duality to the orientifold implies that the same will be true for a threebrane wrapped on $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. Our goal is to understand what the "level two" property means in terms of the threebrane.

In terms of the orientifold, $U(k)$ gauge symmetry appears when k sevenbranes coincide at a point P on the orientifold that is not a fixed point.[6] The threebrane wrapped on $\mathbb{R}^2 \times \widetilde{\mathbf{T}}^2$ intersects the sevenbrane on the two-manifold $\mathbb{R}^2 \times P$. A standard quantization of the $3 \cdot 7$ open strings shows that there are massless chiral fermions, in the fundamental representation of $U(k)$ because there are k sevenbranes, propagating on $\mathbb{R}^2 \times P$. This gives the desired $U(k)$ current algebra (which is with a suitable choice of orientation left-moving since the massless fermions are chiral). The problem is to explain why the current algebra is at level *two*.

The answer depends on an interesting detail about how the T-duality to the orientifold acts on Dirichlet onebranes that are transverse to the $\mathbf{T}^2$. Such a one-brane is mapped to a threebrane that wraps *twice* over $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$, as we will explain presently. Because the wrapping number is two, the local structure near P is actually that of *two* threebranes intersecting transversely with k sevenbranes, as a result of which there are twice as many chiral fermion zero modes, and one gets a level two current algebra.

The origin of the twofold wrapping is that, in the Type I description, a Dirichlet onebrane localized on $\mathbb{R}^2 \times P$, since it lives at a single point $P \in \mathbf{T}^2$, does not "see" the $\mathrm{Spin}(32)/\mathbb{Z}_2$ Wilson lines. The allowed winding numbers of the $1 \cdot 1$ open strings hence are conventional integers, with no half-integral shifts resulting from the lack of vector structure. Hence, after T-duality to the orientifold, one must get a threebrane wrapped on $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ with the property that the allowed momenta for $3 \cdot 3$ open strings are the same as they would be if the underlying Type I model had vector structure.

But as we have by now extensively seen, the $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ that arises when there is no vector structure is of one-half the usual size. So the quantum of momentum for a threebrane wrapped once on $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ would be *twice* what we want. The cure for this involves a mechanism that we have already used, in reverse, in discussing the conventional open strings, and which has entered many times in studies of branes and string dualities. Ignoring for a moment the orientifold projection, we start with *two* threebranes wrapping on $\widetilde{\mathbf{T}}^2$. There is a world-volume $U(2)$ gauge symmetry group. We choose a $U(2)/\mathbb{Z}_2$ flat connection on $\widetilde{\mathbf{T}}^2$ such that the monodromies A, B obey $AB = -BA$. This makes sense for the $3 \cdot 3$ open strings since they transform in the adjoint representation of $U(2)$.[7] In the adjoint representation of $U(2)$, the matrices A and B commute and can be simultaneously diagonalized, with the result that the eigenvalues of (A, B) are $(\pm 1, \pm 1)$, with each possible pair occurring precisely once. Since an eigenvalue -1 shifts the corresponding momentum by half a unit, the result of this is that the momentum lattice for the $3 \cdot 3$ open strings is not the conventional momentum lattice Γ of $\widetilde{\mathbf{T}}^2$ but is $\frac{1}{2}\Gamma$. Because $\widetilde{\mathbf{T}}^2$ is half the usual

[6] If they coincide at certain orientifold fixed points, one gets an enhanced $SO(2k)$ gauge symmetry, which is at level two in the heterotic string description as explained in section 2.3, and is at level two in the orientifold description for the reason given below.

[7] All other open strings actually present in the theory also make sense, but there are some exotic details in some cases, for example involving the fact that a string transforming in a $U(2)$ representation that does not descend to $U(2)/\mathbb{Z}_2$ can make sense if its center of mass is localized on a submanifold of $\widetilde{\mathbf{T}}^2$ on which the $U(2)/\mathbb{Z}_2$ bundle is trivial.

size of the dual lattice, Γ is twice the appropriate momentum lattice for the $3 \cdot 3$ open strings, and hence $\frac{1}{2}\Gamma$ is the right lattice.

After the orientifold projection to $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$, the windings and momenta used to classify the states in the last two paragraphs are only conserved modulo two. Nonetheless, a correct T-duality must match up the spectra in the weak coupling limit, making possible the analysis in the last two paragraphs. The remaining point is that the orientifold projection on the threebranes must be such that the Wilson lines A, B exist. This orientifold projection breaks $U(2)$ to $SO(2)$ (since Type I D-strings have SO and not Sp Chan-Paton factors) and must conjugate A and B to A^{-1} and B^{-1} (since the $\mathbb{Z}_2$ acts as -1 on $\widetilde{\mathbf{T}}^2$). An orientifold projection of the SO kind admits matrices A, B with the claimed properties.

3.3 Compactification to lower dimensions

In this subsection, we consider an issue that is of some interest, though outside the main theme of the present paper: compactification of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ superstring on a torus $\mathbf{T}^n$ with $n > 2$.

The possible $\mathrm{Spin}(32)/\mathbb{Z}_2$ bundles are classified by the characteristic class $\widetilde{w}_2 \in H^2(\mathbf{T}^n, \mathbb{Z}_2)$. $\widetilde{w}_2$ can be regarded as a second rank antisymmetric tensor of the group $SL(n, \mathbb{Z}_2)$. Just like an ordinary real-valued second rank antisymmetric tensor, it can by the action of $SL(n, \mathbb{Z}_2)$ be brought to a skew-diagonal form in which the only non-zero matrix elements are the $1 \cdot 2$, $3 \cdot 4$, $\ldots$, $(2k-1) \cdot 2k$ components, for some k with $2k \leq n$. k is thus the only invariant of $\widetilde{w}_2$, and up to the action of $SL(n, \mathbb{Z})$ (the mapping class group of $\mathbf{T}^n$, which by reduction modulo two induces the action of $SL(n, \mathbb{Z}_2)$ on $\widetilde{w}_2$), the topological type of the bundle is determined by k.

There is thus up to diffeomorphism only one type of non-trivial bundle on $\mathbf{T}^2$ or $\mathbf{T}^3$, but on $\mathbf{T}^4$ a second case appears, with $k = 2$, and on $\mathbf{T}^6$, there is a third possibility with $k = 3$. The $k = 2$ bundle on $\mathbf{T}^4$ is characterized by $\int_{\mathbf{T}^4} \widetilde{w}_2^2 \neq 0$ (where integration is understood as a pairing in mod two cohomology), and for $k = 3$ one likewise has $\int_{\mathbf{T}^6} \widetilde{w}_2^3 \neq 0$.

Let us briefly examine the compactification on $\mathbf{T}^4$ with $k = 2$. A flat connection on a $\mathrm{Spin}(32)/\mathbb{Z}_2$ bundle X with $\widetilde{w}_2^2 \neq 0$ is characterized, in a suitable coordinate system, by $SO(32)$-valued Wilson lines U, V, A, B (the holonomies around the four directions in $\mathbf{T}^4$), with $UV = -VU$, $AB = -BA$, while U and V commute with A and B. Based on our experience in section 2, we can readily construct examples of such matrices in several ways:

(1) We embed $SU(2) \times SU(2)' \times SO(8)$ (here $SU(2)$ and $SU(2)'$ are simply two copies of $SU(2)$) in $SO(32)$, in such a way that the vector of $SO(32)$ decomposes as $\mathbf{2} \otimes \mathbf{2}' \otimes \mathbf{8}$, where $\mathbf{2}$, $\mathbf{2}'$, and $\mathbf{8}$ are the standard representations of the three factors. Then we take $U = u \times 1 \times 1$, $V = v \times 1 \times 1$, $A = 1 \times u \times 1$, $B = 1 \times v \times 1$, with u and v taken from eqn. (2.2). This gives a flat connection on the bundle X with unbroken symmetry an $SO(8)$ group at level 4.

(2) We embed $SU(2) \times O(2) \times Sp(4)$ in $SO(32)$ in such a way that the vector of $SO(32)$ decomposes as $\mathbf{2} \times \mathbf{2}' \times \mathbf{8}$, where $\mathbf{2}$, $\mathbf{2}'$, and $\mathbf{8}$ are again the standard representations of the three factors. Then we take $U = u \times 1 \times 1$, $V = v \times 1 \times 1$ with u and v as in eqn. (2.2); and we take $A = 1 \times u \times 1$, $B = 1 \times v \times 1$, with now u and v taken from eqn. (2.3). This gives a flat connection on the bundle X with the unbroken gauge symmetry being an $Sp(4)$ group at level 2. This construction has $6 = 2 \times 3$ variants, since the pair U, V can be exchanged with A, B, and also we recall from section 2.1 that the construction in eqn. (2.3) has three variants.

(3) Finally, we embed $O(2) \times O(2)' \times SO(8)$ in $SO(32)$ (here $O(2)$ and $O(2)'$ are simply two copies of $O(2)$) in $SO(32)$, in such a way that the vector of $SO(32)$ decomposes as $\mathbf{2} \otimes \mathbf{2}' \otimes \mathbf{8}$, where $\mathbf{2}$, $\mathbf{2}'$, and $\mathbf{8}$ are the standard representations of the three factors. Then we take $U = u \times 1 \times 1$, $V = v \times 1 \times 1$, $A = 1 \times u \times 1$, $B = 1 \times v \times 1$, with u and v taken from eqn. (2.3). This gives a flat connection on the bundle X with unbroken symmetry an $SO(8)$ group at level 4. There are $9 = 3 \times 3$ variants of this construction, since the construction in eqn. (2.3), which we used twice, has three variants.

Adding the above, we see we have in all 10 constructions of flat bundles with $SO(8)$ gauge symmetry, and 6 with $Sp(4)$ gauge symmetry. Along the lines of arguments in section 2, one can show that these 16 bundles lie in one connected component of the moduli space of flat connections on X. Moreover, this component is parametrized by the positions of four fivebrane pairs on the orientifold $\widetilde{\mathbf{T}}^4/\mathbb{Z}_2$. Enhanced gauge symmetry arises when all fivebranes are at one of the 16 orientifold fixed points. 10 of the fixed points are of $\mathcal{O}^+$ type and produce SO gauge groups, while 6 are of $\mathcal{O}^-$ type and produce Sp gauge groups. The SO and Sp orientifold points have respectively a fivebrane charge -2 or $+2$, so the net fivebrane charge is -8, and four fivebrane pairs is the right number to ensure overall neutrality. Moreover, T-duality of the $k = 2$ model leads, along the lines of arguments that we have seen for $k = 1$, to this orientifold with 10 $\mathcal{O}^+$ and 6 $\mathcal{O}^-$ fixed points.

Similarly, the $k = 3$ model is related to a $\widetilde{\mathbf{T}}^6/\mathbb{Z}_2$ orientifold with 36 $\mathcal{O}^+$ fixed points of threebrane charge $-1/2$, 28 $\mathcal{O}^-$ fixed points of threebrane charge $+1/2$, and two threebrane pairs. When all threebranes approach a $\mathcal{O}^+$ point, one gets $SO(4)$ gauge symmetry at level 8, and when all threebranes approach a $\mathcal{O}^-$ point, one gets $Sp(2)$ gauge theory at level 4.

The $k = 2$ and $k = 3$ models have gauge groups of rank reduced by 12 and 14, respectively, compared to standard toroidal compactification. They appear to correspond (as suggested by Y. Oz and also in [8]) to the second and third models in the CHL series, as enumerated in the introduction to [17].

Components of the same topological type

To complete and clarify the picture, and as preparation for the appendix, we should perhaps add the following. For bundles of a particular topological type on $\mathbf{T}^n$ with $n > 2$, there can be several different components of the moduli space of flat connections and hence several different components of the moduli space of superstring vacua. So the number of models is more than one would get from the topological classification alone.

Here is a simple example for $n = 3$ and a topologically trivial bundle. Consider a $\mathbf{T}^3/\mathbb{Z}_2$ orientifold with a $\mathbb{Z}_2$-invariant configuration of 32 sixbranes on $\mathbf{T}^3$, with an odd number of sixbranes at each of the eight orientifold planes. It can be shown that this is T-dual to a superstring compactification on $\mathbf{T}^3$ with an $SO(32)$ bundle that obeys $w_1 = w_2 = w_3 = 0$.[8] These are sufficient conditions for an $SO(32)$ bundle in three dimensions to be topologically trivial. So the model is connected topologically to the standard $SO(32)$ model with the trivial flat connection on a trivial bundle. However, it is not connected to the trivial flat connection via a family of flat connections; it is contained in a component of the moduli space of flat connections in which there are non-simply-laced unbroken symmetry groups, such as $SO(25)$ (at level one), while the trivial flat connection is in a component of the moduli space of flat connections in which all unbroken symmetry groups are simply-laced.

There is no analog of this phenomenon on $\mathbf{T}^2$. By using the complex structure of $\mathbf{T}^2$, one can prove (as in a footnote in section 2.2), that for any given semi-simple gauge group G (regardless of whether $\pi_1(G)$ vanishes), the moduli spaces of flat connections on a bundle of a given topological type are all connected and irreducible.

For sufficiently large n, there are different components of the moduli space of flat connections on a topologically non-trivial $\mathrm{Spin}(32)/\mathbb{Z}_2$ bundle on $\mathbf{T}^n$, just as we saw above for topologically trivial bundles on $\mathbf{T}^3$. To construct such bundles in the orientifold language, one simply places odd numbers of branes at orientifold points of type $\mathcal{O}^+$ (the ones that give SO gauge symmetry) in such a way that $w_1 = w_2 = 0$. These last restrictions (which cannot be obeyed in the case $n = 2$) are needed to get a $\mathrm{Spin}(32)/\mathbb{Z}_2$ model, as opposed to an $O(32)/\mathbb{Z}_2$ or $SO(32)/\mathbb{Z}_2$ model that could not make sense for the $\mathrm{Spin}(32)/\mathbb{Z}_2$ superstring.

4 Threebrane probes and F-theory

4.1 Fivebranes and classical gauge theory

In this section, we consider Type I fivebranes wrapped on $\mathbf{T}^2$ and their T-dual, which will involve threebrane probes of the $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ orientifold. By following a familiar logic [12, 13], we will deduce from the behavior of the threebrane probe an F-theory description of the orientifold.

[8] This was demonstrated as follows by E. Sharpe. In computing the mod two cohomology classes w_1 and w_2, pairs of orientifolds will not contribute, so one can consider the case of just eight sevenbranes, one at each fixed point. This corresponds to an $SO(8)$ bundle whose total Stieffel-Whitney class is $\prod_{i=1}^{8}(1 + x_i)$, where x_i run over all eight elements of $H^1(\mathbf{T}^3, \mathbb{Z}_2)$. One can compute that this product equals 1, modulo two and modulo four-dimensional classes (which vanish as we are in three dimensions). Likewise, an $SO(7)$ bundle that is the direct sum of the seven non-trivial real line bundles of order two over $\mathbf{T}^3$ has total Stieffel-Whitney class $\prod_{i=1}^{7}(1 + x_i) = 1$, where now the x_i run over the seven non-zero elements of $H^1(\mathbf{T}^3, \mathbb{Z}_2)$, and is topologically trivial. This assertion will be important in the appendix.

A system of k parallel Type I fivebranes has a world-volume gauge symmetry whose Lie algebra is that of $Sp(k)$ [24]. There are hypermultiplets that transform under $Sp(k) \times SO(32)$ in the representation $\mathbf{2k} \otimes \mathbf{32}$, where the two factors are the basic representations of $Sp(k)$ and $SO(32)$, respectively. (There also are hypermultiplets transforming in the traceless antisymmetric tensor of $Sp(k)$.) In compactification without vector structure, there is a mod two obstruction to the existence of particles transforming in the $\mathbf{32}$ of $SO(32)$. In order for the $\mathbf{2k} \otimes \mathbf{32}$ to make sense, there must therefore be an equal (and canceling) obstruction to the existence of the $\mathbf{2k}$.

In our problem of compactification on $\mathbb{R}^8 \times \mathbf{T}^2$ without vector structure, we wish to consider k parallel fivebranes whose world-volumes will be $\mathbb{R}^4 \times \mathbf{T}^2$, where $\mathbb{R}^4$ is a subspace of $\mathbb{R}^8$. From what has been said above, the fivebrane world-volume gauge group is effectively really $Sp(k)/\mathbb{Z}_2$, with no "symplectic structure" on the $\mathbf{T}^2$. Hence a supersymmetric state of the fivebrane system should be described by a flat connection whose holonomies B and C (along the two directions of $\mathbf{T}^2$) commute in $Sp(k)/\mathbb{Z}_2$, but if lifted to $Sp(k)$ obey

$$BC = -CB. \tag{4.1}$$

The analysis described in section 2 for $\mathrm{Spin}(4n)/\mathbb{Z}_2$ flat connections without vector structure has a very close analog for $Sp(k)/\mathbb{Z}_2$ flat connections without symplectic structure. As in section 2, an important role is played by certain special examples of such flat connections with maximal unbroken symmetry:

(1) Picking a subgroup $Sp(1) \times O(k)$ of $Sp(k)$ (under which the $\mathbf{2k}$ of $Sp(k)$ transforms as $\mathbf{2} \otimes \mathbf{k}$), we take $B = b \times 1$, $C = c \times 1$, where b, c are elements of $SU(2) = Sp(1)$ with $bc = -cb$. This gives a flat connection, unique up to gauge transformation, with unbroken $O(k)$.

(2) For k even, picking a subgroup $O(2) \times Sp(k/2)$ of $Sp(k)$ (under which the $\mathbf{2k}$ of $Sp(k)$ transforms as $\mathbf{2} \otimes \mathbf{k}$), we take $B = b \times 1$, $C = c \times 1$, where b, c are elements of $O(2)$ with $bc = -cb$. This gives a flat connection with unbroken $Sp(k/2)$; as we saw in section 2, there are three choices for b, c, up to conjugation. If k is odd, a slight variant of this construction gives unbroken $Sp((k-1)/2)$.

Arguments just like those in section 2 for the $SO(4n)/\mathbb{Z}_2$ case show (in keeping with the general theorem mentioned in a footnote in section 2.2) that the four flat connections just described are contained in one component of the moduli space of flat $Sp(k)/\mathbb{Z}_2$ connections on $\mathbf{T}^2$. Moreover, the moduli of these flat bundles can be described in terms of the motion of a $\mathbb{Z}_2$-invariant configuration of k threebranes on an orientifold $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. For reasons very similar to those seen in section 2, the $\widetilde{\mathbf{T}}^2$ that appears here is *not* the usual dual torus (which parametrizes flat $U(1)$ connections on the original $\mathbf{T}^2$) but has half the size. The case considered in (1) above is the case of k threebranes at the $\mathcal{O}^-$ orientifold plane, which we recall gives Sp gauge symmetry for sevenbranes but which evidently gives SO for threebranes. Conversely, case (2) is the case of k threebranes at one of the three $\mathcal{O}^+$ orientifold planes, which give SO gauge symmetry for sevenbranes but evidently give Sp gauge symmetry for threebranes. When k is even, instead of speaking of a $\mathbb{Z}_2$-invariant configuration of k threebranes on $\widetilde{\mathbf{T}}^2$, we can speak of $k/2$ threebranes on

$\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. When k is odd, there is a single threebrane "stuck" at the $\mathcal{O}^-$ point plus $(k-1)/2$ pairs, so one cannot quite reduce the discussion to branes on the quotient $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. The restriction to even k in (2) above reflects the fact that for odd k, one cannot move all threebranes to a $\mathcal{O}^+$ point.

Instead of proving all of these statements by writing out formulas similar to those in section 2, it seems more illuminating to give direct arguments for the first cases $k = 1, 2$; these will actually suffice for our applications. For $k = 1$, we have $Sp(1) = SU(2)$ and group elements B, C with $BC = -CB$ are uniquely determined up to conjugation. This is the case of one threebrane stuck at $\mathcal{O}^-$, with no moduli.

For $k = 2$, which is the first instance in which one really sees the full structure, we embed $Sp(1) \times SO(2) \subset Sp(2)$ (with as usual the $\mathbf{4}$ of $Sp(2)$ transforming as $\mathbf{2} \times \mathbf{2}$) and write $B = b \times b'$, $C = c \times c'$ (here $b, c \in Sp(1)$ and $b', c' \in SO(2)$), with $bc = -cb$ and $b'c' = c'b'$, so b and c are uniquely determined up to conjugation and

$$
\begin{aligned}
b' &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
c' &= \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix},
\end{aligned}
\tag{4.2}
$$

for some θ, ψ. This gives a surjective map from $SO(2)$ flat connections to $Sp(2)/\mathbb{Z}_2$ flat connections without symplectic structure, but there are some discrete identifications, as in section 2. To see these and to elucidate the structure of the moduli space, we use the fact that $Sp(2)/\mathbb{Z}_2 = SO(5)$. In $SO(5)$, B and C become in a suitable basis the commuting matrices

$$
\begin{aligned}
B &= \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \cos 2\theta & \sin 2\theta \\ 0 & 0 & 0 & -\sin 2\theta & \cos 2\theta \end{pmatrix} \\
C &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \cos 2\psi & \sin 2\psi \\ 0 & 0 & 0 & -\sin 2\psi & \cos 2\psi \end{pmatrix}.
\end{aligned}
\tag{4.3}
$$

The fact that these formulas depend trigonometrically on 2θ and 2ψ shows that there are symmetries $\theta \to \theta + \pi$ and (independently) $\psi \to \psi + \pi$, as a result of which the torus $\widetilde{\mathbf{T}}^2$ appearing in the orientifold has one-half the usual size. From (4.3), it is straightforward to see that the generic unbroken gauge symmetry is $U(1)$,[9] and that the unbroken symmetry

[9] Just as in section 2, we will somewhat imprecisely use the name "unbroken gauge symmetry" to refer to the subgroup of $Sp(2)/\mathbb{Z}_2$ consisting of group elements which if lifted to $Sp(2)$ commute with B and C. Including also those that anticommute with B or C gives an additional $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by matrices $\mathrm{diag}(\pm1, \pm1, \pm1, 1, 1)$ of determinant one. These are present for all values of θ, ψ, commute with $U(1)$ and its extensions described below, and are related to the fact that the winding lattice of the dual torus has one-half the usual size.

is enhanced in precisely the following two cases: (1′) If $\theta = \psi = 0$, an extra $\mathbb{Z}_2$ appears (generated by the matrix $\mathrm{diag}(-1,-1,-1,-1,1)$) and the unbroken symmetry is $O(2)$ rather than $SO(2) = U(1)$. (2′) If 2θ and 2ψ are both 0 or π (and not both zero) there is an unbroken $SO(3) = Sp(1)/\mathbb{Z}_2$. (For example, if 2ψ and 2θ are both π, then the unbroken $SO(3)$ acts in the $3 \cdot 4 \cdot 5$ subspace in the basis used in (4.3).) These two cases are just the specialization to $k = 2$ of the special cases (1), (2) of $Sp(k)/\mathbb{Z}_2$ flat connections listed earlier. The modulus $a = (2\theta, 2\psi)$ up to the $SO(5)$ Weyl transformation $a \to -a$ parametrizes the position of a threebrane on the orientifold $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$.

The fact that for $k = 2$ – two fivebranes on $\mathbf{T}^2$ – we get a dual description with only *one* threebrane on $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ is, of course, another manifestation of the curious factors of two that have appeared throughout this paper.

4.2 F-theory interpretation

Now our goal is to give an F-theory interpretation of $\mathbf{T}^2$ compactification without vector structure. At first sight, this poses a conundrum, since the F-theory description must involve F-theory on a Calabi-Yau two-fold, but there is no obvious suitable candidate. F-theory on K3 is the heterotic string on $\mathbf{T}^2$ with vector structure, and F-theory on $\mathbf{T}^2 \times \mathbf{T}^2$ is simply Type IIB on $\mathbf{T}^2$.

The answer that we will find is that $\mathbf{T}^2$ compactification without vector structure is dual to F-theory on a K3 which is constrained to have a novel kind of D_8 singularity. This D_8 singularity looks macroscopically like an ordinary D_8 singularity (at long distances from it there is a standard D_8 ALE space), but has in a mysterious way absorbed some sort of flux, such that it cannot be blown up or deformed away and does not generate gauge symmetry. If however this D_8 is extended to a D_{8+m} singularity, then $Sp(m)$ gauge symmetry appears.

To obtain this result, we simply examine as in [13] the theory on a threebrane probe. We consider as in section 4.1 a pair of fivebranes wrapped on $\mathbf{T}^2$, dual to a single threebrane probe on $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. The "base space" $B = \widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ is as a complex manifold isomorphic to $\mathbf{CP}^1$. On the threebrane probe, there is a $U(1)$ gauge field whose τ parameter determines up to isomorphism an elliptic curve that varies holomorphically with the position of the probe on B. By determining the structure of the resulting elliptic fibration $X \to B$, one obtains an F-theory description. As already suggested, X will be a K3 surface with a "frozen" D_8 singularity.

The underlying theory of two fivebranes and 32 ninebranes has gauge symmetry $Sp(2) \times SO(32)$, with hypermultiplets transforming as $\mathbf{4} \otimes \mathbf{32}$ (the tensor product of the fundamental representations of the two groups) plus $\mathbf{5} \otimes \mathbf{1}$, where the $\mathbf{5}$ is the traceless antisymmetric tensor of $Sp(2)$ or equivalently the vector of $SO(5)$.

In the dual description on $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$, for a generic threebrane position, the unbroken gauge symmetry is a $U(1)$ that appears as the second factor in $Sp(1) \times U(1) \subset Sp(1) \times O(2) \subset Sp(2)$. (Here as before, the $\mathbf{4}$ of $Sp(2)$ is $\mathbf{2} \otimes \mathbf{2}$ of $Sp(1) \times O(2)$, fixing the embeddings in the chain just mentioned.) In the $\mathbf{4}$ of $Sp(2)$, the unbroken $U(1)$ has charges 1 and -1, while in the **5** (as it arises in the tensor product of two **4**'s) the charges are 2, -2, and 0. The factor of 2 in the charges of the **5** relative to the **4** will be important. (This factor has actually already appeared in (4.3).)

The details of the elliptic fibration over B depend on where we place the eight sevenbranes. We place them in generic, distinct points. Now in the motion of the threebrane, there are 12 exceptional cases at which monodromy occurs. The threebrane may collide with one of the eight sevenbranes, or with one of the four orientifold planes. There are three essential cases:

(1) If the threebrane meets a sevenbrane, a single charge one hypermultiplet becomes massless. (In the Type I description, it originates in the $\mathbf{4} \otimes \mathbf{32}$, which is why it has charge 1.) This gives according to [25] a monodromy conjugate to

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.4}$$

In all we get eight such points from the eight sevenbranes.

(2) If the threebrane meets an orientifold plane of type $\mathcal{O}^+$, one gets at the classical level an enhanced $SO(3)$ gauge symmetry with no massless hypermultiplets. At the quantum level [25], this splits into a pair of quantum singularities of monodromy T. From the three $\mathcal{O}^+$ planes, we get $2 \times 3 = 6$ such points.

(3) The remaining case is that the threebrane meets an orientifold plane of type $\mathcal{O}^-$. In this case, at the classical level, the $U(1)$ gauge symmetry of the threebrane is enhanced to $O(2)$. Two hypermultiplets, of charge 2, become massless at this point. They are components of the **5** of $SO(5)$ that correspond to the two eigenvalues $B = C = 1$ that can be seen in eqn. (4.3) at $\theta = \psi = 0$, and have charge two because, as explained above, the charged components of the **5** have charge ± 2 (or equivalently, because of the factor of 2 multiplying θ and ψ in (4.3)). These give a monodromy conjugate to $-T^4$, as we will compute momentarily.

To verify the assertion that the monodromy is $-T^4$, we note that in $U(1)$ gauge theory with several hypermultiplets of charge q_i and zero bare mass, the monodromy around the singular point at the origin of the Coulomb branch (where $a = a_D = 0$ and the hypermultiplets become massless) is conjugate to T^Δ with $\Delta = \sum_i q_i^2$. In the present case, with two hypermultiplets of charge 2, this would give $\Delta = 8$. However, we actually have $O(2)$ and not $SO(2)$ gauge symmetry, which simply means that we must divide by a discrete transformation that maps $(a, a_D) \to (-a, -a_D)$. This gives an extra factor of -1 in the monodromy. In addition, since the monodromy we want corresponds to going only "half-way" around the origin in the complex a-plane, the exponent Δ is replaced by $\Delta/2$. So the monodromy for gauge group $O(2)$ is not conjugate to T^8, as it would be for $U(1)$, but to $-T^4$.

An elliptic fibration with section that has a singular fiber of monodromy conjugate to $-T^4$ has, in its Weierstrass model, a singularity of type D_8 on that fiber. So the elliptic fibration $X \to B$ that describes $\mathbf{T}^2$ compactification without vector structure has a D_8 singularity. It also has, counting the results of (1) and (2) above, $8 + 6 = 14$ fibers with generic singularities (nodes or ordinary double points) with monodromy conjugate to T. Ordinarily, a D_8 singular fiber can be deformed to 10 generic singular fibers, which in the present context would give an elliptic fibration $X \to B$ with $14+10=24$ generic singular fibers. Such an X would have Euler characteristic 24, and would be a K3 surface. In the present context, evidently, as there are only eight sevenbranes, one is not permitted to deform away the D_8 singularity. Evidently, $\mathbf{T}^2$ compactification without vector structure is dual to F-theory on an elliptic K3 surface X with an irremovable D_8 singularity.

If m of the 8 sevenbranes on $B = \mathbf{P}^1$ approach the $\mathcal{O}^-$ point, then according to the analysis in section 2, we get an enhanced $Sp(m)$ gauge symmetry, at level one. On the other hand, moving m sevenbranes to a D_8 fiber enhances the singularity to D_{8+m}. Hence, we conclude that an F-theory D_8 singularity of this type has the property that if it is enhanced to D_{8+m}, a level one $Sp(m)$ gauge symmetry is generated in spacetime.

4.3 Reduction to M-theory

Now we would like to discuss what happens when one compactifies further, to go to an M-theory description.

Since Type I on $\mathbf{T}^2$ without vector structure is equivalent to F-theory on a K3 surface X with a D_8 singularity, one might think that Type I on $\mathbf{S}^1 \times \mathbf{T}^2$, with an obstruction to vector structure on the $\mathbf{T}^2$, would be equivalent to M-theory on the same surface X. It seems, however, that this is not quite so.

If we start with Type I on $\mathbf{S}^1 \times \mathbf{T}^2 = \mathbf{T}^3$, and perform T-duality, we get an orientifold $(\widetilde{\mathbf{S}}^1 \times \widetilde{\mathbf{T}}^2)/\mathbb{Z}_2$ with eight orientifold sixplanes and eight sixbrane pairs. (The factors $\widetilde{\mathbf{S}}^1$ and $\widetilde{\mathbf{T}}^2$ are of course the duals to the factors $\mathbf{S}^1$ and $\mathbf{T}^2$ in the original $\mathbf{T}^3$). Two of the orientifold planes are of type $\mathcal{O}^-$, and six are of type $\mathcal{O}^+$. A quick way to verify these statements is to look at the maximal unbroken symmetry groups. On $\mathbf{T}^2$ without vector structure, a maximal symmetry group was $Sp(8)$ and it occurred in a unique fashion up to gauge transformation. On $\mathbf{S}^1 \times \mathbf{T}^2$, with an obstruction to vector structure that comes from the second factor, $Sp(8)$ is still a maximal symmetry group, but it occurs in *two* ways up to gauge transformation. (1) One can "pull back" a flat connection with $Sp(8)$ symmetry from the second factor in $\mathbf{S}^1 \times \mathbf{T}^2$. (2) One can also "twist" it by a Wilson line on the $\mathbf{S}^1$ factor with holonomy in the center of $Sp(8)$; as this center is $\mathbb{Z}_2$, this gives precisely one new possibility. Likewise, the second maximal symmetry group, which is $O(16)$, appears three times on $\mathbf{T}^2$ but six times on $\mathbf{S}^1 \times \mathbf{T}^2$. This counting of flat connections with maximal symmetry shows that the numbers of $\mathcal{O}^-$ and $\mathcal{O}^+$ sixplanes are two and six. The rank of the possible symmetry groups shows that the number of sixbrane pairs is eight. One can also verify vanishing of the net sixbrane charge. The two kinds of orientifold plane have charge $+4$ and -4, and one has $2 \cdot 4 - 6 \cdot 4 + 2 \cdot 8 = 0$.

So this model corresponds to a Type IIA orientifold on $(\widetilde{\mathbf{T}}^3/\mathbb{Z}_2)$ with two $\mathcal{O}^-$ sixplanes and six $\mathcal{O}^+$ sixplanes. We expect the model to have an M-theory description in terms of compactification on a (perhaps singular) K3 surface because it was obtained by compactifying on an extra circle from a model that had such an F-theory description.

In M-theory, a sixplane of type $\mathcal{O}^+$ does not correspond to any singularity [26, 27, 28]. To account for the fact that $\mathbf{S}^1 \times \mathbf{T}^2$ compactification of Type I without vector structure gives eight fewer vector multiplets than the usual case, each of the two $\mathcal{O}^-$ planes must lead to a singularity of the K3 surface that reduces the rank of the gauge group by four. By analogy with what we found in the F-theory case, the most obvious possibility is that an $\mathcal{O}^-$ sixplane is converted in M-theory to a rank four $A-D-E$ singularity that does not produce gauge symmetry.

In fact, by showing that this hypothesis makes it possible to solve certain four-dimensional gauge theories, Landsteiner and Lopez [16] argued fairly convincingly that an $\mathcal{O}^-$ sixplane corresponds to a D_4 singularity which does not generate gauge symmetry. (They expressed the singularity associated with the $\mathcal{O}^-$ sixplane as the quotient of the singular surface $xy = v^4$ by a $\mathbb{Z}_2$ symmetry $x \leftrightarrow y$, $v \leftrightarrow -v$. By introducing the invariant functions $w = v^2$, $a = (x-y)v$, $b = x+y$, which obey $a^2 = wb^2 - 4w^3$, one sees that the quotient is in fact a D_4 singularity.)

We now have what at first looks like a contradiction between the following facts. (1) Type I on $\mathbf{T}^2$ without vector structure gives F-theory on a K3 surface X with a D_8 singularity that does not give gauge symmetry. (2) F-theory on $\mathbf{S}^1 \times X$, for any X, is equivalent to M-theory on X. (3) Type I on $\mathbf{S}^1 \times \mathbf{T}^2$, with an obstruction to vector structure coming from the $\mathbf{T}^2$, is equivalent to M-theory on X with two D_4 singularities that do not give gauge symmetry. The apparent contradiction is that according to (1), it seems that the Type I model on $\mathbf{S}^1 \times \mathbf{T}^2$ under consideration here should give M-theory on a K3 with a D_8 singularity, but according to (3), the actual singularity is $D_4 \times D_4$.

The resolution of the issue apparently has to do with the following subtlety about F-theory, which affects the precise meaning of statement (2). In F-theory, one really only has a variable τ parameter, with certain singularities, over a base B. One then interprets this geometrically in terms of an elliptic fibration given in a Weierstrass model. (Given the τ parameter as a function on B, one has the data to canonically construct a Weierstrass model by writing the equation $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$ which can be written once τ is known. One does not generally have the information to canonically write other models, but under certain conditions, for instance if X is singular, other models obeying a Calabi-Yau condition do exist.) For example, when we assert that Type I on $\mathbf{T}^2$, without vector structure, is equivalent to F-theory on a K3 surface with a D_8 singularity, we really mean that the monodromy of the τ parameter is such that the Weierstrass model of the corresponding elliptic fibration has a D_8 singularity.

After compactifying on an additional circle and going to M-theory, the fibers of the elliptic fibration become "physical" and one gets an actual geometrical K3 surface. Evidently, in the case of $\mathbf{T}^2$ compactification without vector structure, the model that results after compactifying on an additional circle is not the Weierstrass model, with its D_8 singularity, but a different model of the same elliptic fibration, with two D_4 singularities. I will not explain why this is so, but I will show that the second model does exist.

By blowing up a D_8 singularity in a complex surface, one gets a configuration of eight genus zero curves arranged according to the Dynkin diagram of D_8. (Each of the eight genus zero curves corresponds to a node in the D_8 Dynkin diagram, and two nodes in the Dynkin diagram are connected if and only if the corresponding curves intersect.) If the D_8 singularity is contained in a fiber of an elliptic fibration, then the fiber itself is a ninth genus zero curve (the generic fiber of an elliptic fibration has genus one, but fibers containing singularities have genus zero). The nine curves are arranged according to the Dynkin diagram of $\widehat{D}_8$ (the affine extension of D_8).

Given a K3 surface that contains a configuration of nine genus zero curves arranged according to the $\widehat{D}_8$ Dynkin diagram, one can produce a variety of birational models (which are all singular Calabi-Yau manifolds) by "blowing down" some of them to produce a singularity. The blown down curves make up a subdiagram of the $\widehat{D}_8$ diagram, and this subdiagram determines the type of singularity. For instance, in the Weierstrass model, eight curves forming a D_8 subdiagram of the $\widehat{D}_8$ diagram are blown down, giving a D_8 singularity. This is the model of a singular K3 surface X that can be used for an F-theory description of Type I on $\mathbf{T}^2$ without vector structure.

In general, one can blow down any proper subset of the nine curves (but not all nine) to get a singular Calabi-Yau manifold that is birational to X. For example, if one omits the "middle" node of the $\widehat{D}_8$ diagram, one is left with the Dynkin diagram of $D_4 \times D_4$. By blowing up the curve that corresponds to the "middle" node, and blowing down all others, one gets a K3 surface Y that is birational to X and has two D_4 singularities. From what has been said above, M theory on Y is equivalent to Type I on $\mathbf{S}^1 \times \mathbf{T}^2$ without vector structure. The D_4 singularities have of course, like the D_8 singularity in the analogous F-theory version, the property that they do not generate gauge symmetry and cannot be resolved or deformed away.

5 Duality to CHL model

$\mathbf{T}^2$ compactification without vector structure has at least one more dual description. It is equivalent to a form of the CHL model, in fact, to compactification of the $E_8 \times E_8$ heterotic string on $\mathbf{T}^2$ with the two E_8's swapped in going around one circle of the $\mathbf{T}^2$. This has been somewhat implicitly sketched in [17], and derived more fully in [7]. After giving a direct explanation of why this is true, we will give a more theoretical explanation, which will also show that within a certain class of constructions, there are no more models of the same kind.

In fact, we can construct either the CHL model or $\mathbf{T}^2$ compactification without vector structure via an involution (that is, a $\mathbb{Z}_2$ symmetry) of a Narain lattice $\Gamma^{17,1}$, as follows:

(A) We write $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$ and call the two factors the "first" and "second" $\mathbf{S}^1$, respectively. To build the CHL model, we consider conventional $\mathbf{S}^1$ compactification of the $E_8 \times E_8$ heterotic string. This is described by a Narain lattice $\Gamma^{17,1}$, with a particular decomposition into left- and right-movers. Picking a decomposition $\Gamma^{17,1} = \Gamma^8 \oplus \Gamma^8 \oplus \Gamma^{1,1}$, where the Γ^8's are copies of the root lattice of E_8, we define an involution w of $\Gamma^{17,1}$ that exchanges the two Γ^8's (breaking $E_8 \times E_8$ to a diagonal E_8) and acts trivially on $\Gamma^{1,1}$. To get the CHL model, we compactify on the second circle, with a monodromy around the second circle consisting of the automorphism w of the Narain lattice.

(B) We pick $\mathrm{Spin}(32)/\mathbb{Z}_2$ matrices U, V with $UV = -VU$. We compactify the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string (which of course is equivalent to the Type I heterotic string) on the first circle with a Wilson line U. This gives a model that can be described by a Narain lattice $\Gamma^{17,1}$, with a particular decomposition into left- and right-movers, and which is in particular equivalent (up to motion in moduli space) to the first step in (A) above. The $\mathrm{Spin}(32)/\mathbb{Z}_2$ matrix V gives an automorphism of this theory which is an involution $\widetilde{w}$ of the Narain lattice. Now, to obtain $\mathbf{T}^2$ compactification without vector structure, we compactify on the second circle in $\mathbf{T}^2$, with a monodromy around the second circle consisting of the automorphism $\widetilde{w}$ of the Narain lattice.

Note, that, if we wish, we can in either (A) or (B) take Wilson lines along the first circle (corresponding in case (A) to Wilson lines in the diagonal E_8 and in case (B) to a suitable choice of U) for which the unbroken gauge group is abelian. In this case, there is no unbroken $SU(2)$ and the vacuum is determined by a vector v that does not lie on any of the walls introduced below. This fact will be important later. Also, the ability to pick convenient Wilson lines makes possible the following very direct explanation of why the two models are the same.

In (A), we start with a large radius R on the first circle, and a Wilson line on it that breaks $E_8 \times E_8$ to $SO(16) \times SO(16)$. There is of course a monodromy around the second circle that exchanges the two E_8's and therefore the two $SO(16)$'s. Now we interpolate to small R. The $E_8 \times E_8$ heterotic string, in its vacuum with unbroken $SO(16) \times SO(16)$, on a circle of radius R, is T-dual to the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string, on a circle of radius $1/R$, likewise in a vacuum with unbroken $SO(16) \times SO(16)$. Of course, if R is small, the $\mathrm{Spin}(32)/\mathbb{Z}_2$ description is more useful. In this description, a Wilson line around the first circle breaks $\mathrm{Spin}(32)/\mathbb{Z}_2$ to $SO(16) \times SO(16)$. The Wilson line in question is our friend $U = u \times 1$, where u is the $O(2)$ matrix introduced in eqn. (2.3). The monodromy around the second circle that exchanges the two $SO(16)$'s becomes in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ description a Wilson line $V = v \times 1$, with v as in (2.3). So we have arrived at the familiar Wilson lines $U = u \times 1$, $V = v \times 1$, on $\mathbf{T}^2$, showing that model (A) is T-dual to model (B).

The relation of the $E_8 \times E_8$ heterotic string to M-theory gives a natural explanation for this result; we will explain it briefly. The $E_8 \times E_8$ model in ten dimensions is equivalent to M-theory on an interval $\mathbf{S}^1/\mathbb{Z}_2$. Model (A) is obtained by compactifying further on $\mathbf{S}^1 \times \mathbf{S}^1$ in such a way that in going around the first $\mathbf{S}^1$, the two end-points of $\mathbf{S}^1/\mathbb{Z}_2$ are exchanged. We write the resulting three-manifold with boundary as $(\mathbf{S}^1/\mathbb{Z}_2 \times_W \mathbf{S}^1) \times \mathbf{S}^1$, where W is the "flip" that exchanges the two ends of $\mathbf{S}^1/\mathbb{Z}_2$. (Thus, $\mathbf{S}^1/\mathbb{Z}_2 \times_W \mathbf{S}_1$ is a Möbius strip; the equivalence of the CHL model to M-theory on a Möbius strip has been noted in [29, 30].) On the other hand, M-theory on $(\mathbf{S}^1/\mathbb{Z}_2 \times_W \mathbf{S}^1) \times \mathbf{S}^1$ is equivalent to Type IIA on $\mathbf{S}^1/\mathbb{Z}_2 \times_W \mathbf{S}^1$. We note now that Type IIA on $\mathbf{S}^1/\mathbb{Z}_2$ should be interpreted [9, 10] as a T-dual of Type I on $\mathbf{S}^1$ with $SO(32)$ broken to $SO(16) \times SO(16)$ and an unbroken $SO(16)$ at each fixed point; in this Type I description, W becomes a gauge transformation that exchanges the two $SO(16)$'s. Fibering now over $\mathbf{S}^1$ with monodromy W generates the familiar Type I model without vector structure on $\mathbf{S}^1 \times \mathbf{S}^1$.

Geometry of the T-duality group

We now want to show that, in a sense, models (A) and (B) "must" be the same, since there is only one model of their kind. For this, we must explain something about the geometry of T-duality for the heterotic string in nine dimensions. The duality group is not quite the full symmetry group $O(17,1;\mathbb{Z})$ of the lattice $\Gamma^{17,1}$, because one is limited to transformations that do not reverse the direction of "time," or in other words do not change the sign of the right-moving momentum. (Transformations that reverse the sign of the right-moving momentum do not respect the GSO projection of the heterotic string.) The subgroup of $O(17,1;\mathbb{Z})$ that preserves the direction of time will be called $F = O_+(17,1;\mathbb{Z})$. The Narain moduli space $\mathcal{M} = O_+(17,1;\mathbb{Z})\backslash O_+(17,1;\mathbb{R})/O(17;\mathbb{R})$ can be constructed in two steps. The quotient $H = O_+(17,1;\mathbb{R})/O(17;\mathbb{R})$ is the hyperboloid that parametrizes timelike future-pointing vectors v of length squared -1 in $\mathbb{R}^{17,1}$. Physically, in the application to the heterotic string, v generates the subspace of $\mathbb{R}^{17,1}$ that (at a given point in the moduli space) consists of right-moving momenta. The moduli space $\mathcal{M}$ of vacua is the quotient $F\backslash H$. So it is useful to understand the action of F on H.

For every vector $m \in \Gamma^{17,1}$ of length squared 2, there is an elementary reflection R_m in $O_+(17,1;\mathbb{Z})$, which acts by $v \to v - m(m,v)$. The condition for v to be a fixed point of this transformation is $(m,v) = 0$, which means that in the heterotic string vacuum determined by v, m is purely left-moving and so is the highest root of an enhanced $SU(2)$ gauge symmetry (in which R_m is a Weyl transformation), which is unbroken when $(m,v) = 0$. The condition $(m,v) = 0$ defines a hyperplane T_m in H which we will call a wall.

According to theorem 1 of [31], the subgroup F' of F that is generated by the reflections R_m for various m is of index two. The F' action on H can be fairly conveniently described. The walls T_m divide H into regions which (as can be shown via general arguments about reflection groups) are fundamental domains for the action of F'. Each such fundamental domain is bounded by nineteen walls, which correspond to points m_i, $i = 1, \ldots, 19$, of length squared two. (Each fundamental domain goes to infinity in two ways, corresponding to unbroken $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$.) The m_i are called the simple positive roots of $\Gamma^{17,1}$ (for the given choice of fundamental domain). Their matrix of inner products determines a Dynkin (or Coxeter) diagram, with 19 nodes, that is associated with the lattice $\Gamma^{17,1}$. It is drawn on p. 529 of [31], and looks like two copies of the extended Dynkin diagram of E_8 (a total of $2 \times 9 = 18$ nodes) combined by attaching the extended nodes at the end of the $\widehat{E}_8$ diagrams to a nineteenth node. This Dynkin diagram has precisely one diagram automorphism Υ, which exchanges the two E_8's.[10] This diagram automorphism generates the group $F/F' \cong \mathbb{Z}_2$ of symmetries of $\Gamma^{17,1}$ that

[10]The same diagram automorphism also acts as a diagram automorphism of a $\widehat{D}_{16}$ diagram that is a subdiagram of the $\Gamma^{17,1}$ diagram. The fact that Υ induces diagram automorphisms of either $\widehat{E}_8 \times \widehat{E}_8$ or

cannot be obtained as products of reflections. As we will see, it is responsible for the existence of models (A) and (B).

In building models along the lines of (A) or (B), the idea is to first compactify on a circle $\mathbf{S}^1$ with a vacuum determined by some point in H that is invariant under some $x \in F$. Because of this H-invariance, it is possible, in compactifying on a second circle, to obtain a "twisted" model by saying that the physics is rotated by x in going around the second circle. So in analyzing the possible monodromies of twisted models, we can restrict ourselves to x's that have fixed points in H.

A case that (as suggested by J. H. Conway) is quite pertinent and can be analyzed very quickly is the case that x leaves fixed a point in the interior of a fundamental domain $\mathcal{F}$. In this case, x clearly permutes among themselves the 19 boundary "walls" of $\mathcal{F}$, and therefore permutes the 19 simple roots m_i that determine these walls. Since x is an element of $O_+(17,1;\mathbb{Z})$, in permuting the m_i it preserves the geometric relations among them. So x acts as an automorphism of the Dynkin diagram, which, if not the identity, must be Υ. The m_i generate $\Gamma^{17,1}$, so x is determined by its action on them. There is therefore (other than the identity) only one element of F with a fixed point in the interior of $\mathcal{F}$, namely Υ.

This suffices for comparing models (A) and (B), since (as noted earlier) both models are constructed using monodromies that leave invariant points in H at which the unbroken gauge group is abelian. Such points do not lie on any wall but rather lie in the interior of a fundamental domain. For a given fundamental domain, there is only one choice of an element of F (other than the identity) that leaves fixed a point in its interior. So, after conjugating by an element of F' that identifies the fundamental domains used in (A) and (B), these models are constructed with the same monodromy element and hence coincide.

Furthermore, because of the uniqueness of Υ, this is the only model that can be built using a monodromy that leaves fixed a vacuum on the first circle in which the unbroken gauge group is abelian. Can any new models be constructed using monodromies that leave fixed only vacua with nonabelian gauge symmetry? By arguments similar to the ones used to show that the walls divide H into fundamental domains, it can be shown that any model with monodromy in F' is equivalent to standard toroidal compactification. (One shows that if an element $x \in F'$ leaves fixed a vacuum corresponding to a vector $v \in H$, then x is an element of the Weyl group of the unbroken gauge group G of this vacuum. The monodromy x can be continuously rotated to the identity in G, showing that the model twisted by x is equivalent to standard $\mathbf{T}^2$ compactification.) This leaves the question, which will not be resolved here, of whether new models can be constructed using a monodromy x that is not in F' and leaves fixed only vacua with nonabelian gauge symmetry. (Considerations of supergravity show that, if so, the twisting by x in going around the second circle breaks the nonabelian gauge symmetry, which will therefore not actually be present in eight dimensions.)

$\widehat{D}_{16}$ subdiagrams can possibly serve as the starting point for an alternative explanation of the relation between constructions (A) and (B).

6 The third orientifold

6.1 F-theory description and T-duality

Apart from the most familiar case discussed in [9, 10] and the additional case that we have already considered, there is one more supersymmetric orientifold with target $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$. Any such orientifold has n_+ sevenplanes of type $\mathcal{O}^+$, and n_- of type $\mathcal{O}^-$, with $n_+ + n_- = 4$. The models with $(n_+, n_-) = (4, 0)$ and $(3, 1)$ are by now familiar; what remains is the model with $(n_+, n_-) = (2, 2)$. (A supersymmetric orientifold on $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ cannot have $n_- > 2$, because in that case the net sevenbrane charge of the orientifolds would be positive, and could not be canceled except by adding anti-sevenbranes and violating supersymmetry.) The main purpose of the present section is to understand some dual descriptions of the $(n_+, n_-) = (2, 2)$ model.

F-theory dual

In fact, we can immediately identify an F-theory dual. It will be a K3 elliptic fibration over $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2 \cong \mathbf{P}^1$ with the following structure. An $\mathcal{O}^+$ sevenplane splits nonperturbatively into a pair of fibers with generic (nodal or normal crossing) singularities [12], and we saw in section 4 that an $\mathcal{O}^-$ sevenplane is represented in F-theory as a D_8 singularity that, for presently obscure reasons, cannot be resolved or deformed. So the $(n_+, n_-) = (2, 2)$ model has an F-theory description via a K3 fibration over $\mathbf{P}^1$ with two of these exotic D_8 singularities and four generic singular fibers.

Search for a T-dual

Our remaining attention will therefore be directed at seeking a T-dual of the $(n_+, n_-) = (2, 2)$ model. It is most natural, however, to first consider a similar question that arises in compactification to *nine* dimensions. There are two Type IIA orientifolds on $\mathbf{S}^1/\mathbb{Z}_2$. One model, with $(n_+, n_-) = (2, 0)$, and 16 eightbranes (or 16 eightbrane pairs on the covering space $\mathbf{S}^1$), has been extensively studied. The second possibility is $(n_+, n_-) = (1, 1)$. In this case, the net eightbrane charge of the orientifolds vanishes, so there are no eightbranes. We will seek a T-dual of this model, that governs its behavior in the limit that the radius of the $\mathbf{S}^1/\mathbb{Z}_2$ vanishes. In seeking this T-dual, we will be guided by the following clues:

(1) The $\mathbf{RP}^2$ contribution to the vacuum amplitude vanishes, because of a cancellation between the contributions from the two orientifold planes. In fact, there is a symmetry of the free closed string sector – the one that flips the two ends of $\mathbf{S}^1/\mathbb{Z}_2$ – under which the $\mathbf{RP}^2$ path integral is odd. However, there are unoriented world-sheets in the theory, and amplitudes that violate the symmetry just mentioned receive non-zero contributions from $\mathbf{RP}^2$. For future use, note that if $\mathbf{S}^1$ is parametrized by an angle θ with $0 \leq \theta \leq 2\pi$, then the symmetry that exchanges the two ends of $\mathbf{S}^1/\mathbb{Z}_2$ is $\theta \to \theta + \pi$, which acts on a state of momentum p along the circle as $(-1)^p$; so the $\mathbf{RP}^2$ amplitude violates conservation of $(-1)^p$.

(2) In the weak coupling limit, the spectrum is the same as that of the closed string sector of the $(1, 1)$ orientifold on $\mathbf{S}^1/\mathbb{Z}_2$, but the interactions are different. By T-duality,

the spectrum is also the same, in the weak coupling limit, as that of a Type I superstring on $\mathbf{S}^1$. This strongly suggests that the target space of the T-dual of the $(1,1)$ orientifold should be $\mathbf{S}^1$; its spectrum on an $\mathbf{S}^1$ of large radius should match the spectrum of the $(1,1)$ orientifold for small radius.

6.2 The T-dual

We will now describe a simple model that obeys the required properties.[11]

We start with a Type IIB-like model[12] with target space $\mathbf{S}^1$. We permit unoriented worldsheets, but only in a way that is correlated with the map to $\mathbf{S}^1$. In fact, we require that the orientation of the worldsheet Σ is reversed in going around a loop $\gamma \in \Sigma$ if and only if γ wraps around $\mathbf{S}^1$ an odd number of times. A succinct way to say this is as follows. We let α be the generator of $H^1(\mathbf{S}^1, \mathbb{Z}_2)$. We permit only maps $\Phi : \Sigma \to \mathbf{S}^1$ such that $\Phi^*(\alpha) = w_1(\Sigma)$. Here $w_1(\Sigma)$ (the first Stieffel-Whitney class of Σ) is the one-dimensional cohomology class that assigns the value -1 to a closed loop on Σ if the orientation of Σ is reversed in going around Σ and $+1$ otherwise.

In this model, the $\mathbf{RP}^2$ contribution to the vacuum amplitude vanishes, for the following reason. $\mathbf{RP}^2$ can be obtained from a disc D by imposing a certain equivalence relation on the boundary of D. If the boundary, which is of course a circle, is parametrized by an angular variable ψ, with $0 \leq \psi \leq 2\pi$, then one can take the equivalence relation to be $\psi \to \psi + \pi$. Since the orientation of $\mathbf{RP^2}$ is reversed in going from $\psi = 0$ to $\psi = \pi$, the restriction to maps Φ with $\Phi^*(\alpha) = w_1(\Sigma)$ means that we must consider only maps Φ such that, when ψ is increased from 0 to π, one wraps $2n+1$ times around the target $\mathbf{S}^1$, for some integer n. Therefore, the boundary of D – the full region $0 \leq \psi \leq 2\pi$ – is wrapped by Φ a total of $4n+2$ times around the target $\mathbf{S}^1$. In particular, the wrapping number is always nonzero, so such a Φ cannot be extended over all of D, that is over $\mathbf{RP}^2$. This confirms that the $\mathbf{RP}^2$ contribution to the vacuum amplitude vanishes.

To what physical processes can $\mathbf{RP}^2$ contribute? Clearly, it can contribute only to the expectation value of a product of vertex operators for physical states whose total winding number is of the form $4n+2$ for some integer n. We will see momentarily that all physical states in this theory have even winding number. So if we call the winding number w, then the free closed string sector has a $\mathbb{Z}_2$ symmetry that multiplies a state of winding w by $(-1)^{w/2}$. The fact that all amplitudes on $\mathbf{RP}^2$ violate conservation of winding number by $4n+2$ units for some integer n means that the $\mathbf{RP}^2$ contribution is odd under $(-1)^{w/2}$. We compare this to the fact that, as explained at the end of the last subsection, the $\mathbf{RP}^2$ amplitude of the $(1,1)$ orientifold is odd under $(-1)^p$. We are led to suspect that there is a duality between these theories that maps $w/2$ to p; as in our previous experience, the factor of 2 corresponds to an unusual factor of 2 in the radius of the T-dual to the

[11] This model was described in a different but equivalent language, along with many of the properties discussed below, in section 3.1 of [29]. In a notation we will introduce presently, the model was described there as a Type IIB orientifold with $M = \mathbb{R}^9 \times \mathbf{S}^1$ and τ_M a π rotation of $\mathbf{S}^1$.

[12] By this we mean simply a closed string model with left- and right-moving world-sheet supersymmetry and the same GSO projection for left- and right-movers.

$(1,1)$ orientifold. (In the $(1,1)$ orientifold and its dual, the quantities that we have called p and $w/2$ are only defined mod 2, because of the existence of unoriented worldsheets in one case and an orientifold projection in the other.)

Now let us analyze the spectrum of this model. For this we work in a Hamiltonian description. We take Σ to be the product of a circle C, on which we quantize our strings, times a "time" direction, parametrized by τ, $0 \leq \tau \leq T$ for some T. In gluing $\tau = T$ to $\tau = 0$, one may either reverse or not reverse the orientation of C, so that the worldsheet is either a torus or a Klein bottle. However, in going around C, the orientation of Σ is not reversed.

Because of that last fact, the restriction $\Phi^*(\alpha) = w_1(\Sigma)$ means that C must wrap an *even* number of times around the target space $\mathbf{S}^1$. Thus, the winding number w is even, as was asserted earlier.

Let R be the radius of the target $\mathbf{S}^1$. The fact that only even windings are allowed in this model suggests that its spectrum may have a simple comparison to a more standard string model on a circle of radius $2R$. We will now see this by examining the windings in the τ direction to determine the spectrum of momenta.

For a conventional oriented closed string mapped to a circle of radius R, the sum over momentum states comes from the following factors in the worldsheet path integral. There is a factor of R for the center of mass of the worldsheet, and also a sum over windings in the time direction. Together, after a Poisson summation, these factors give a sum over momentum states of the string of the form

$$\sum_{n\in\mathbb{Z}} q^{n^2/R^2}, \tag{6.1}$$

where q depends on the length of the torus (or Klein bottle) in the time direction.

Now in the present problem, because $\Phi^*(\alpha) = w_1(\Sigma)$, the sum over windings in the time direction is modified. Only even windings are allowed if the worldsheet is a torus, and only odd windings are allowed if the worldsheet is a Klein bottle.

Let us first think about the torus contribution. Having only even windings has roughly the same effect as doubling the radius of the target space, which would double the basic allowed unit of winding. But a factor of 2 is missing, since in our problem one gets a factor of R from the string center of mass position, while a conventional closed string on a circle of radius $2R$ would get a factor of $2R$ from the center of mass position. So the contribution of the even time windings (that is the even windings in the time direction) can be obtained from (6.1) by replacing R by $2R$ and dividing by 2, and so is

$$\frac{1}{2}\sum_{n\in\mathbb{Z}} q^{n^2/(2R)^2}. \tag{6.2}$$

The sum over odd time windings is of course the difference between the sum (6.1) over all windings and the sum (6.2) over even windings. In our problem, the sum over the odd windings is accompanied by an operation Ω that reverses the worldsheet orientation, so we represent this contribution symbolically by Ω times the difference between (6.1) and

(6.2) or

$$\frac{\Omega}{2}\sum_{n\in 2\mathbb{Z}} q^{n^2/(2R)^2} - \frac{\Omega}{2}\sum_{n\in 2\mathbb{Z}+1} q^{n^2/(2R)^2}. \tag{6.3}$$

The sum of (6.2) and (6.3) can be written

$$\frac{1+\Omega}{2}\sum_{n\in 2\mathbb{Z}} q^{n^2/(2R)^2} + \frac{1-\Omega}{2}\sum_{n\in 2\mathbb{Z}+1} q^{n^2/(2R)^2}. \tag{6.4}$$

Only by the minus sign in the last term does this differ from the conventional momentum sum and orientation projection of a standard Type I string on a circle of radius $2R$. That sum would read, in the same notation,

$$\frac{1+\Omega}{2}\sum_{n\in \mathbb{Z}} q^{n^2/(2R)^2}. \tag{6.5}$$

Comparing the last two formulas, we see that the present model, on a circle of radius R, has for even momentum precisely the same states as a conventional Type I superstring on a circle of radius $2R$. For odd momentum, the model considered here has an orientation projection $(1-\Omega)/2$ while the standard Type I superstring has an orientation projection $(1+\Omega)/2$. However, since Ω changes the sign of the momentum, the masses and quantum numbers that one gets for states of non-zero momentum, and therefore in particular for states of odd momentum, are the same whether one projects onto $\Omega = 1$ or onto $\Omega = -1$. Consequently, the present model on a circle of radius R has the same spectrum as the ordinary Type I string on a circle of radius $2R$, but the detailed wavefunctions of physical states are different, and consequently the interactions will be different. As summarized at the end of section 6.1, this is the expected behavior for the T-dual that we are seeking of the $(1,1)$ orientifold on $\mathbf{S}^1/\mathbb{Z}_2$.

Systematic perturbation expansion of the orientifold

To be more precise in comparing the model just described to the $(1,1)$ orientifold on $\mathbf{S}^1/\mathbb{Z}_2$, we need a precise recipe for systematic worldsheet perturbation theory on the orientifold. We describe this for a general orientifold that does not have an open string sector (one could also elaborate the description to incorporate the open strings), and then specialize to the case that the target space is $\mathbb{R}^9 \times \mathbf{S}^1$.

We consider a spacetime M with an involution τ_M (an involution is simply a symmetry whose square is the identity). The Riemann surfaces in the theory will be orientable Riemann surfaces Σ with orientation-reversing freely-acting involution τ_Σ. (One allows the case in which Σ consists of two identical components exchanged by Σ, as a result of which the quotient Σ/τ_Σ is orientable but not endowed with an orientation, and also the case in which Σ is connected and the quotient Σ/τ_Σ is unorientable.) The worldsheet path integral of this theory will be taken over maps $\Phi : \Sigma \to M$ which commute with the τ's in the sense that $\Phi \circ \tau_\Sigma = \tau_M \circ \Phi$. (Such an equivariant formalism was used in [32].)

The most standard worldsheet action of this theory would be the usual Nambu-Goto action

$$I = \frac{1}{2}\frac{1}{4\pi\alpha'}\int_\Sigma \sqrt{g}g^{ab}\partial_a X^I \partial_b X^J G_{IJ}. \tag{6.6}$$

The factor of $\frac{1}{2}$ is included so as to normalize the action in the standard way for maps from Σ/τ_M rather than for maps from Σ. The path integral of the theory will run over τ-invariant maps, with an integrand that we could take to be simply e^{-I}. We want to generalize this, to obtain theories with orientifold planes of different kinds.

To do this, we take an arbitrary element $x \in H^1_{\mathbb{Z}_2}(M, \mathbb{Z}_2)$, the $\mathbb{Z}_2$-equivariant cohomology of M with $\mathbb{Z}_2$ coefficients; here $\mathbb{Z}_2$ is the group generated by τ_M. Then $\Phi^*(x)$ is an element of $H^1_{\mathbf{Z}_2}(\Sigma, \mathbb{Z}_2)$, where now $\mathbb{Z}_2$ is generated by τ_Σ. Since $\mathbb{Z}_2$ acts freely on Σ, this is the same as $H^1(\Sigma/\mathbb{Z}_2, \mathbb{Z}_2)$. The obstruction $w_1(\Sigma/\mathbb{Z}_2)$ to orientation of Σ/τ_Σ also takes values in $H^1(\Sigma/\mathbb{Z}_2, \mathbb{Z}_2)$, and there is a well-defined mod 2 pairing $(w_1(\Sigma/\mathbb{Z}_2), \Phi^*(x))$. We can incorporate this pairing in the definition of the worldsheet Lagrangian and take the integrand to be

$$e^{-I}(-1)^{(w_1(\Sigma/\mathbb{Z}_2),\Phi^*(x))}. \tag{6.7}$$

The advantage of this somewhat abstract definition is that, because of its cohomological nature, it is manifest that all factorization or cut and paste requirements of string theory are satisfied. Hence one does get a systematic string perturbation theory for any x.

We will now specialize to the case that M is $\mathbb{R}^9 \times \mathbf{S}^1$, with $\mathbf{S}^1$ parametrized by an angle θ, and τ_M is $\theta \to -\theta$. The goal is to show that for suitable x, we do get the desired $(1,1)$ orientifold with one fixed point of type $\mathcal{O}^-$ and one of type $\mathcal{O}^+$. To do this, we take x to be the cohomology class dual to a $\mathbb{Z}_2$-invariant cycle in $M = \mathbb{R}^9 \times \mathbf{S}^1$ consisting of one of the orientifold planes, say the one at $\theta = 0$. Now we consider the case that $\Sigma = \mathbf{S}^2$ and $\Sigma' = \Sigma/\mathbb{Z}_2$ is $\mathbf{RP}^2$. If Φ is a constant map from Σ to the orientifold plane at $\theta = \pi$ (which induces a constant map from $\mathbf{RP}^2$ to that orientifold plane), then $\Phi^*(x) = 0$, because the support of x is disjoint from $\theta = \pi$. If, however, Φ is a constant map from $\mathbf{RP}^2$ to the orientifold plane at $\theta = 0$, then the normal bundle to the orientifold plane pulls back to the "orientation bundle" (the determinant of the tangent bundle) of $\mathbf{RP}^2$. In this case, $\Phi^*(x) = w_1(\mathbf{RP}^2)$ and is in particular non-zero. Also, $(w_1(\mathbf{RP}^2), \Phi^*(x)) = w_1(\mathbf{RP}^2)^2 = 1$ modulo 2. So the sign factor in (6.7) is -1 for this map, confirming that the $\theta = 0$ orientifold is of opposite type from the $\theta = \pi$ orientifold.

I will not attempt to give in this paper a formal proof that the orientifold with x as in the last paragraph is T-dual to the model of section 6.2. However, T-duality brings about a kind of Fourier transform, and would be expected to map a sign factor $(-1)^{(w_1(\Sigma/\mathbb{Z}_2),\Phi^*(x))}$ to a delta function setting $w_1(\Sigma/\mathbb{Z}_2)$ equal to $\Phi^*(x)$. This delta function was used in the definition of the model of section 6.2.

Answer to the original question

At this point, we can answer the original question, which was to describe a T-dual of the $(2,2)$ orientifold on $\mathbf{T}^2/\mathbb{Z}_2$. I claim that this is obtained simply by compactifying on a circle the model of section 6.2 which is T-dual to the $(1,1)$ orientifold on $\mathbf{S}^1/\mathbb{Z}_2$.

The $(2,2)$ orientifold on $\mathbf{T}^2/\mathbb{Z}_2$ can be described as follows in the language of the present discussion. Take spacetime to be $M = \mathbb{R}^8 \times \mathbf{T}^2$. Let τ_M be the usual involution that acts as -1 on $\mathbf{T}^2$. Now consider an orientifold on M/τ_M with the following choice of x. If $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$, and θ is an angular parameter on the first $\mathbf{S}^1$, let x be the Poincaré dual to the τ-invariant hypersurface $\theta = 0$. Consider the orientifold model with this choice of x and the action as in (6.7). An analysis as in the last subsection shows that the two orientifold planes at $\theta = 0$ are of one type and the two at $\theta = \pi$ are of the other type, so this model is the $(2,2)$ orientifold.

On the other hand, T-duality will turn $\mathbf{T}^2/\mathbb{Z}_2$ into a dual $\mathbf{T}^2$, and will Fourier transform the sign factor $(-1)^{(w_1(\Sigma/\mathbb{Z}_2),\Phi^*(x))}$ to a delta function setting $w_1(\Sigma/\mathbb{Z}_2)$ equal to $\Phi^*(x)$. The model on $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$ with this delta function is simply the compactification on $\mathbf{S}^1$ (the $\mathbf{S}^1$ in question being the second factor of $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$) of the model that we introduced and analyzed in section 6.2 and found to be T-dual to the $(1,1)$ orientifold on $\mathbf{S}^1/\mathbb{Z}_2$.

Systematic perturbative description of the $(3,1)$ *orientifold*

We have given a systematic and manifestly factorized description of the worldsheet perturbation expansion for the $(1,1)$ orientifold in nine dimensions, and the $(2,2)$ orientifold in eight dimensions. At this point, one might like to describe the eight-dimensional $(3,1)$ orientifold, which has been of course the main subject of the present paper, in an analogous way. We will do so, at least for the closed string sector.

For this, we return first to the general formulation of orientifold perturbation theory, in terms of maps from a Riemann surface Σ with a free, orientation-reversing involution τ_Σ to a spacetime M that is equipped with an involution τ_M. The path integral is over maps $\Phi : \Sigma \to M$ such that $\tau_\Sigma \circ \Phi = \Phi \circ \tau_M$. Suppose that we are given an element $y \in H^2_{\mathbb{Z}_2}(M, \mathbb{Z}_2)$. The pullback $\Phi^*(y)$ is an element of $H^2_{\mathbb{Z}_2}(\Sigma, \mathbb{Z}_2)$, which since τ_Σ acts freely on Σ is the same as $H^2(\Sigma/\mathbb{Z}_2, \mathbb{Z}_2)$. So there is a well-defined sign factor

$$(-1)^{\int_{\Sigma/\mathbb{Z}_2} \Phi^*(y)}. \tag{6.8}$$

Because of its cohomological nature, its inclusion in the path integrand is manifestly compatible with all factorization requirements of string theory.

Let us implement this procedure for the case that $M = \mathbb{R}^8 \times \widetilde{\mathbf{T}}^2$, with $\widetilde{\mathbf{T}}^2$ the "dual torus" parametrized by angles θ, ψ. We take τ_M to be the involution $\theta \to -\theta$, $\psi \to -\psi$, with the usual four orientifold fixed points at which θ and ψ are both 0 or π. We take y to be the Poincaré dual to the $\mathbb{Z}_2$-invariant cycle $\theta = \psi = 0$. I claim that in this case, inclusion of the factor (6.8) in the path integrand has the effect of reversing the "type" of the orientifold plane at $\theta = \psi = 0$ without affecting the others, so that it generates the closed string sector of the $(3,1)$ orientifold from that of the standard $(4,0)$ orientifold.

For this it suffices to show that, if $\Sigma = \mathbf{S}^2$, $\Sigma/\tau_\Sigma = \mathbf{RP}^2$, with Φ a constant map to an orientifold fixed point, then the sign factor in (6.8) is -1 if $\theta = \psi = 0$ and otherwise $+1$. The sign factor is $+1$ for $(\theta, \psi) \neq (0,0)$ since the orientifold fixed points away from the origin are disjoint from the support of y. On the other hand, for Φ a constant map to $\theta = \psi = 0$, the pullback of the tangent bundle of $\widetilde{\mathbf{T}}^2$ to $\mathbf{RP}^2$ is the tangent bundle of $\mathbf{RP}^2$, so in this case the sign factor in (6.8) reduces to $(-1)^{w_2(\mathbf{RP}^2)} = -1$.

For $\widetilde{\mathbf{T}}^1/\mathbb{Z}_2$ and $\widetilde{\mathbf{T}}^2/\mathbb{Z}_2$ orientifolds, the cohomological formulas we have described enable one to develop a systematic worldsheet perturbation expansion with an arbitrary labeling of fixed points as being of type $\mathcal{O}^+$ or $\mathcal{O}^-$. (In some cases one will run into trouble because of noncancellation of brane charges, but the formal rules make sense and are compatible with factorization.) For $\widetilde{\mathbf{T}}^n/\mathbb{Z}_2$ with $n > 2$, it does not appear that an arbitrary assignment of the "type" of the orientifold planes will give a theory that can be described in a systematic string perturbation expansion. For restrictions on the orientifold configuration in an analogous situation, see [33].

A A note on four-dimensional Gauge theories

In this appendix we suggest a resolution to a longstanding puzzle concerning the computation of the supersymmetric index $\mathrm{Tr}\,(-1)^F$ in supersymmetric gauge theories in four dimensions. As in much of the body of the paper, the key point will be to understand certain facts about the components of the moduli space of flat $\mathrm{Spin}(n)$ connections on a torus.

Consider four dimensional supersymmetric Yang-Mills theory, with a connected, simple, and simply-connected[13] gauge group G and no chiral superfields. This model has a discrete chiral symmetry group $\mathbb{Z}_{2h}$, where h is the dual Coxeter number.

It is believed that this model undergoes spontaneous chiral symmetry breaking, with $\mathbb{Z}_{2h}$ spontaneously broken to $\mathbb{Z}_2$ (its maximal subgroup that permits gluino bare masses). This results in the existence of h distinct vacua, each of which is believed to have a mass gap (and confinement). If one formulates the theory on $\mathbb{R}^1 \times \mathbf{T}^3$, with $\mathbb{R}^1$ being the "time direction," and $\mathbf{T}^3$ being a spatial three-torus, then one expects a vacuum with a mass gap to contribute $+1$ to $\mathrm{Tr}\,(-1)^F$. (The contribution of a vacuum without a mass gap would not necessarily be $+1$.) One therefore expects $\mathrm{Tr}\,(-1)^F = h$.

On the other hand, an explicit computation of $\mathrm{Tr}\,(-1)^F$ was made in section 8 of [34] by actually counting, for weak coupling, the supersymmetric states on a torus. This was done by first finding the classical moduli space $\mathcal{M}$ of zero energy states on $\mathbf{T}^3$, and then performing a Born-Oppenheimer quantization. The classical states of zero energy are given by representations of the fundamental group of $\mathbf{T}^3$ in G, or in other words by a choice (up to conjugation) of three commuting elements U_1, U_2, and U_3 of G. It was argued that by quantizing the component of $\mathcal{M}$ that contains $U_1 = U_2 = U_3 = 1$, one gets a contribution $r+1$ to $\mathrm{Tr}\,(-1)^F$, where r is the rank of G.

One is thus led to expect that $h = r+1$. This is in fact so for $SU(n)$ (with $h = n$, $r = n-1$), and it is also true for $Sp(n)$ (with $h = n+1$, $r = n$). The puzzle is that it is not so for $\mathrm{Spin}(n)$ with $n \geq 7$ that $h = r+1$. (The $\mathrm{Spin}(n)$ groups with $n = 3, 5$, or 6 do work as they are equivalent to $SU(2)$, $Sp(2)$, and $SU(4)$. $\mathrm{Spin}(4)$ is not simple so the above discussion does not precisely apply to it, but since $\mathrm{Spin}(4) = SU(2) \times SU(2)$, a slightly corrected version of the formula does work for $\mathrm{Spin}(4)$.) On the contrary, for

[13] As in section 7 of [34], one can also consider the case that G is not simply-connected (and/or not connected), in which case one will encounter topologically non-trivial G bundles on $\mathbf{T}^3$. However, to illustrate the essential issues we wish to consider here, it suffices to focus on the case that G is connected and simply-connected, in which case the bundle is topologically trivial.

Spin($2k$) with $k \geq 3$ one has $h = 2k - 2$, $r = k$, and for Spin($2k+1$) with $k \geq 2$ one has $h = 2k - 1$, $r = k$.

The error in [34] was to assume that $\mathcal{M}$ is connected and to evaluate the contribution only of the component of $\mathcal{M}$ containing $U_1 = U_2 = U_3 = 1$, which we will call the trivial flat connection. It is true for $SU(n)$ that $\mathcal{M}$ is connected. In fact, any family of commuting elements of $SU(n)$, such as U_1, U_2, and U_3, can be simultaneously diagonalized, or in other words conjugated to a maximal torus. As the maximal torus is connected and abelian, this means that $\mathcal{M}$ is connected. Likewise any family of commuting elements of $Sp(n)$ can be conjugated to a maximal torus (a proof of this by induction on the number of commuting group elements uses the fact that any one element of $Sp(n)$ has the following properties: *(i)* it can be conjugated to a maximal torus; *(ii)* its commutant is a product of unitary and symplectic groups). So $\mathcal{M}$ is connected for $Sp(n)$. $\mathcal{M}$ is likewise connected for Spin(n) with $n \leq 6$ because of equivalences mentioned in the last paragraph. But this fails for Spin(n) with $n \geq 7$, as we have seen in section 3.3, and this is the reason that for those groups one should not expect h to equal $r + 1$.

The general formula for arbitrary simple, connected, and simply-connected G can, however, be worked out by extending the ideas in [34]. Let $\mathcal{M}_i$ be the connected components of $\mathcal{M}$. Let G_i be a maximal unbroken subgroup of a flat connection determined by a point in $\mathcal{M}_i$. Then, as in [19] and our discussion in section 2 for the $\mathbf{T}^2$ case, $\mathcal{M}_i$ is up to a finite cover (which will not affect the following discussion) the same as the moduli space of triples of commuting elements V_1, V_2, V_3 of G_i that are continuously connected to $V_1 = V_2 = V_3 = 1$. The contribution of $\mathcal{M}_i$ to $\mathrm{Tr}\,(-1)^F$ can be computed by the same computation as in [34], but with G replaced by G_i. Hence, if r_i is the rank of G_i (there may be several possible G_i, as in the analogous case treated in section 2, but they all have the same rank), the contribution of $\mathcal{M}_i$ to $\mathrm{Tr}\,(-1)^F$ is $r_i + 1$.[14]

Summing over all components, the formula saying that the analysis in terms of physical vacua and chiral symmetry breaking should agree with the explicit weak coupling analyis is

$$h = \sum_i (r_i + 1) \tag{A.1}$$

if the G_i are all simple.

Let us verify this formula for Spin(n). We start with Spin(7), which is the first problematical case. Apart from a component $\mathcal{M}_1$ consisting of flat connections that are continuously connected to the identity, the moduli space contains a component $\mathcal{M}_2$ that is an isolated point. The flat connection corresponding to the unique point in $\mathcal{M}_2$ can

[14]This really assumes that the G_i are simple (which will be the case for the examples we consider) since the computation in [34] was for simple groups. Also, the computation in [34] was for connected gauge groups, while in examples below the G_i will not always be connected. In fact, we will meet an example in which one of the G_i is $O(k)$ (or rather its double cover Pin(k)) rather than $SO(k)$. By a reexamination of the argument in [34], one can see that the formula $r+1$ gives the correct result for $O(k)$ for all k, even in the exceptional case $k = 2$. (This would not be so for the non-simple group $SO(2)$, for which $\mathrm{Tr}\,(-1)^F = 0$, so in the discussion below getting the right value of $\mathrm{Tr}\,(-1)^F$ for Spin(9) depends on the fact that the relevant centralizer is $O(2)$ instead of $SO(2)$.)

JHEP02(1998)006

be described, up to conjugation, by commuting holonmies U_1, U_2, U_3, which are diagonal elements with eigenvalues $(U_1, U_2, U_3) = (\pm 1, \pm 1, \pm 1)$, with each of the seven combinations of signs other than $(1, 1, 1)$ appearing with multiplicity one. (As was sketched in a footnote in section 3.3, this bundle has $w_1 = w_2 = 0$, and so is a Spin(7) bundle and in particular is topologically trivial.) This flat connection admits no deformations; its centralizer (the unbroken subgroup of Spin(7)) is a finite group, of rank 0. Meanwhile, the centralizer of $\mathcal{M}_1$ is, of course, Spin(7), of rank 3. That $\mathcal{M}_1$ and $\mathcal{M}_2$ are the only components of $\mathcal{M}$ can be proved using the D-brane description of Spin(7) flat connections; we postpone this argument until the end of the present appendix. So the identity (A.1) becomes $5 = (3 + 1) + (0 + 1)$, compatible with the standard conjectures about the dynamics of the Spin(7) theory.

It is straightforward to generalize this to Spin(n) with $n > 7$. $\mathcal{M}$ has a component $\mathcal{M}_1$ that contains the trivial flat connection; the rank of its centralizer is that of Spin(n). There is as explained below precisely one more component, which parametrizes a family of flat connections of which one example is given by the commuting holonomies $V_i = U_i \oplus \mathbf{1}_{n-7}$, where U_i are the Spin(7) matrices of the last paragraph and $\mathbf{1}_{n-7}$ is an $n-7$-dimensional identity matrix. The rank of the unbroken subgroup is that of Spin($n - 7$).[15] The sum of the rank of Spin(n) and that of Spin($n - 7$) is $r_1 + r_2 = n - 4$, while for Spin(n) one has $h = n - 2$. So we get the expected identity $h = (r_1 + 1) + (r_2 + 1)$, compatible with the standard conjectures concerning the dynamics of the Spin(n) theory.

One expects (A.1) to hold also for the exceptional Lie groups (perhaps with some modifications if the G_i are not simple), but an efficient verification of this really requires a more powerful method of computation.

It remains to verify that for Spin(n) with $n \geq 7$, there are precisely the two components of the moduli space of flat connections claimed above. First we consider $n = 7$. To describe a Spin(7) flat connection, one needs a $\mathbb{Z}_2$-invariant configuration of seven D-branes on $\widetilde{\mathbf{T}}^3$, the covering space of an orientifold $\widetilde{\mathbf{T}}^3/\mathbb{Z}_2$. One of the orientifold planes, call it $\mathcal{O}_0$, corresponds to eigenvalues $(1, 1, 1)$ of U_1, U_2, U_3, and the others, call them $\mathcal{O}_\alpha$, $\alpha = 1, \ldots, 7$, to sequences $(\pm 1, \pm 1, \pm 1)$ with at least one -1. If there are fewer than seven orientifold planes at which the number of zerobranes is odd, then by using the fact that the moduli space of Spin(n) flat connections on $\mathbf{T}^3$ is connected for $n < 7$ (or simply by a direct computation), one shows that either the flat bundle has non-zero w_1 or w_2 and so is not a Spin(7) bundle and is topologically non-trivial (and does not contribute to $\mathrm{Tr}\,(-1)^F$ with untwisted boundary conditions), or it is continuously connected via flat connections to $U_1 = U_2 = U_3 = 1$. So we need only consider the case that seven of the eight orientifold planes contain a single D-brane each. There is hence exactly one orientifold plane that has no D-branes. If it is not $\mathcal{O}_0$, then the bundle has $w_1 \neq 0$ and is again not a Spin(7) bundle. So the only relevant case is the case of precisely one D-brane at each of the $\mathcal{O}_\alpha$, and this gives the flat Spin(7) connection with holonomies U_i that was described earlier.

[15]The unbroken subgroup itself is not Spin($n - 7$) but Pin($n - 7$) (an extension of Spin($n - 7$) to allow orientation-reversing symmetries) times some $\mathbb{Z}_2$'s. That one gets Pin($n - 7$) instead of Spin($n - 7$) is important for $n = 9$, for a reason explained in the last footnote.

For Spin(n) with $n > 7$, one shows by the same argument that the only component of Spin(n) flat bundles on $\mathbf{T}^3$ that is not continuously connected to the trivial flat bundle is obtained as follows. It n is even, one places one D-brane at each orientifold plane and lets the others wander in pairs. If n is odd, one places one D-brane at each of the $\mathcal{O}_\alpha$ and lets the others wander in pairs. In particular, for all $n \geq 7$ there is precisely one component of the moduli space of flat Spin(n) connections apart from the component of the trivial flat connection. An example of a flat connection in this component is obtained (whether n is even or odd) by placing one D-brane at each of the $\mathcal{O}_\alpha$ and $n-7$ at $\mathcal{O}_0$; the corresponding holonomies are $V_i = U_i \oplus \mathbf{1}_{n-7}$ with U_i as above.

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